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Methods of Solving Nonstandard Problems

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ISBN 978-3-319-19886-6 ISBN 978-3-319-19887-3 (eBook)
DOI 10.1007/978-3-319-19887-3

Library of Congress Control Number: 2015945585

Mathematics Subject Classification (2010): 00A07, 26A09, 26A15, 26A48, 26C10, 26D05, 26D07, 97H20, 97H30, 97G60, 08A50, 11B25, 11A05, 11B65, 11C05, 11D25, 11D72, 11J81, 65Hxx, 65H04, 65H05, 65H10

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.springer.com)

*To my beautiful daughter, **Sasha***
*And to my wonderful parents, **Natali** and*
Valery Grigoriev
Your encouragement made this book possible
And to my university mentor and scientific
*advisor academician, **Stepanov Nikolay***
Fedorovich
Without your help and brilliant mind my
career as a scientist would not be successful!

Preface

Many universities offer problem-solving courses for students majoring in mathematics and mathematics teaching. However, I have noticed over the years that many graduates are afraid of solving complex problems and try to escape solving challenging math problems if they can. Why should we be afraid? Our fight or flight response either leads us to take up the struggle with the problems or to avoid mathematics with a passion.

It is not a secret that we tend to like things that we are naturally good at. We love something if we have visible, continuous success. Such success comes only with hard work. When I was young, I myself had difficulties in geometry class, which I described in my book *Methods of Solving Complex Geometry Problems* (Springer, 2013). I later went on to win various Math Olympiads and graduate from Lomonosov Moscow State University (MGU) *summa cum laude*, defend a Ph.D. in Mathematical and Physical Sciences, and publish over 60 papers in the fields of differential equations, game theory, economics, and optimal control theory. Some initial success in mathematics can lead to greater successes over time.

At Moscow State, at the end of each semester the students had to pass four oral exams given by renowned professors. The professors could ask any tricky question on the topic of the examination ticket. When I was preparing for such exams and in order to get an "A," I did not try to memorize all the definitions and proofs, but rather I tried to develop a "global" understanding of the subject. I thought of possible questions that an examiner could ask me and tried to predict the type of problem that I could be asked to solve. I developed my own way of learning and I want to share it with you.

For over 30 years, whenever I spotted an especially interesting or tricky problem, I added it to my notebooks along with my original solutions. I've accumulated thousands of these problems. I use them every day in my teaching and include many of them in this book. Before accepting an academic position at a university, I worked as a teacher at Ursuline Academy of Dallas and used my problem-solving techniques in my students' college preparation. I was pleased to receive appreciation letters from MIT and Harvard where some of my students were admitted.

If you are struggling with math, this book is for you. Most math books start from theoretical facts, give one or two examples, and then a set of problems. In this book, almost every statement is followed by problems. You are not just memorizing a theorem—you apply the knowledge immediately. Upon seeing a similar problem in the homework section, you will be able to recognize and solve it.

Although each section of the book can be studied independently, the book is constructed to reinforce patterns developed at stages throughout the book. This helps you see how math topics are connected. The book can be helpful for self-education, for people who want to do well in math classes, or for those preparing for competitions. The book is also meant for math teachers and college professors who would like to use it as an extra resource in their classroom.

What Is This Book About?

This book will teach you about functions and how properties of functions can be used to solve nonstandard equations, systems of equations, and inequalities. When we say "nonstandard," one can think of a variety of problems that appear unusual, intractable, or complex. However, we can also say that nonstandard can indicate a method that is opposite to a standard or common way of thinking. For example, most of the times we need to find the maximal or minimal value of a function, the standard method would be to use the derivative of a function. However, under certain conditions, maximum and minimum problems can be solved through knowledge of some properties, such as boundedness of functions, and perhaps with the application of known inequalities. Another example of a nonstandard problem would be a word problem whose solution is restricted to the integers or that may be reduced to a nonlinear system with more variables than the number of the equations. A nonstandard problem is one that does not yield easily to direct solution. The nonstandard method of problem solving is the process of synthesizing connections between seemingly disassociated areas of mathematics and selecting appropriate generalizations, so that known constraints coincide to yield the solution. They are the Sudoku puzzles of mathematics.

Standard methods and relevant formulas make up the context for the problem sets and are presented in each chapter together with simple problems for illustration. Basic knowledge of secondary school mathematics is assumed. For example, if a problem is to solve a quadratic equation $x^2 - 7x + 2 = 0$, then its roots, $x_{1,2} = \frac{7 \pm \sqrt{41}}{2}$, can be found using the well-known quadratic formula. What if I change the problem a little bit and ask you now to prove that for a new quadratic equation, $x^2 + ax + 1 - b = 0$ with natural roots, the quantity $a^2 + b^2$ cannot be a prime number? Would standard methods and the quadratic formula help to solve this problem?

The problem has two parameters a and b, and is restricted to the set of natural numbers. Hence, in order to solve this problem, we need to know more than just a quadratic formula; we need to have a method that will provide another constraint of the solution. In this case, for example, Vieta's formula and the knowledge of elementary number theory might be helpful.

Let me give you now the following problem:

Solve the inequality $(x^2 + 2x + 2)^x \geq 1$.

In this problem, we do not have any parameters, but the problem is no less difficult than the previous one. The expression inside parentheses is a quadratic raised to the power of x. Would knowledge of solving standard quadratic or exponential inequalities help here? Do we need to do anything with the unit on the right-hand side? Did you hear anything about monotonic functions in the past that might be helpful?

Let's consider the solution of cubic equations. Many of my students know that the Rational Zero Theorem or the Fundamental Theorem of Algebra might be of help. Some who took a course on the history of mathematics have heard about the Cardano formula. These formulas may be applicable, but they may not be adequate to solve the question.

Many interesting Olympiad problems can be solved by using nonstandard and otherwise nonobvious approaches. For example, what would you do if I ask you to find the value of a parameter a for which the cubic function $f(x) = x^3 - 3x - a$ has precisely two x-intercepts? What condition on a would be necessary so that it would have one or three x-intercepts? Playing with a graphing calculator might give you a hint, but no calculators are allowed in the Mathematics Olympiad.

Knowledge of nonstandard methods of problem solving is important because we develop a deeper understanding of mathematics from these odd questions. Mathematics is not a disjointed collection of topics but rather a unified whole. The connections between fields are what tie them together. The structures of these mathematical fields can be learned in a more powerful way by understanding these connections gained by the exploration of these nonstandard problems.

There is a method for developing solutions to nonstandard problems. After solving some especially interesting problem, look for a generalization and try to see an application of that method in the solution of other problems. Usually, a nonstandard problem requires knowledge of several aspects of mathematics and can be solved only with the knowledge of some particular fact from a seemingly disassociated field.

If students see an elegant solution but do not apply the approach to other problems, they will not remember it, just as nobody remembers phone numbers these days. However, if a teacher uses and reuses the same approach throughout the entire curriculum, students will remember it and learn to value the beauty of the method. This is what I practice in my teaching and share with you in this book.

Let us look at the following problem.

Solve the equation $(x^2 + x - 1)(x^2 + x + 1) = 2$.

If you multiply the two quantities on the left and subtract two from each side, then you would obtain a polynomial equation of the fourth degree $x^4 + 2x^3 + x^2 - 3 = 0$ that cannot be solved by using the Rational Zero Theorem because it does not have integer roots. However, it has two real irrational roots. The substitution $t = x^2 + x$ can simplify the equation and make it solvable.

We might quickly forget this fact if we don't consider a generalization:

Find real solutions of $(x + 1)(x + 2)(x + 3)(x + 4) - 4 = 0$.

There may seem to be nothing in common between the two problems. However, with some experience, you will see that they are similar if instead of multiplying all terms together, we multiply the middle two and outer two pairwise to obtain the equation $(x^2 + 5x + 4)(x^2 + 5x + 6) - 4 = 0$. Then, using the substitution $y = x^2 + 5x + 4$, we rewrite the original equation into a quadratic in a new variable: $y(y + 2) = 0$. The solutions are the irrational roots, $x_{1,2} = \frac{-5 \pm \sqrt{5+4\sqrt{5}}}{2}$.

Further, for any natural n, the expression $n(n + 1)(n + 2)(n + 3) + 1$ is a perfect square since multiplying the two middle terms and two outer terms pairwise we get $(n^2 + 3n + 2)(n^2 + 3n) + 1$, so $(n^2 + 3n)^2 + 2(n^2 + 3n) + 1^2 = (n^2 + 3n + 1)^2 = m^2$.

This book covers very important topics in algebra and analysis with applications. For example, knowledge of bounded functions will allow you to solve many interesting problems like this:

Find all real solutions of the equation $2^{1-|x|} = 1 + x^2 + \frac{1}{1+x^2}$.

This book also explains how to use the boundedness of functions to solve complex problems concerning maxima and minima without using a derivative.

The book covers many theoretical aspects, such as the inequalities of Cauchy-Bunyakovsky and Bernoulli as well as other parameterized relations. You will be able to decide whether $200!$ or 100^{200} is greater and to analytically solve the following problem:

Find all values of a parameter a, for which the equation $a + \sqrt{6x - x^2 - 8} = 3 + \sqrt{1 + 2ax - a^2 - x^2}$ has precisely one solution.

The techniques used in this book are not new, but they are basic to understanding functions. For example, the Babylonians created tables of cubes and squares of the natural numbers and even wrote the solution for any cubic equation of the form of $n^3 + n^2 = m$ on clay tablets. Versions of problems solved by the ancients often reappear in modern math contests. Their importance to modern mathematics is fundamental and unavoidable.

This book is not a textbook. Some knowledge of algebra and trigonometry such as what is introduced in secondary school is necessary to make full use of the material. However, a mastery of these subjects is not a prerequisite. You will use your knowledge of secondary school mathematics in order to better delve into the analysis of functions and their properties as you develop problem-solving skills and your overall mathematical abilities.

The book is divided into four chapters: Solving Problems using Properties of Functions, Polynomials, Problems from Trigonometry, and Unusual and Nonstandard Problems. Each chapter has its own homework. However, there are overlaps in knowledge and concepts between chapters. These overlaps are unavoidable since the threads of deduction we follow from the central ideas of the chapters are intertwined well within our scope of interest. For example, trigonometric functions are the topic of Chapter 3, at which point we also like to discuss the properties of some parameterized trigonometric functions. However, a full exposition of parameterized functions is not taken up until Chapter 4. For the same reason, we will on occasion use the results of a particular lemma or theorem in a solution but wait to prove that lemma or theorem until it becomes essential to the thread at hand. If you know that property, you can follow along right away and, if not, then you may find it in the following sections or in the suggested references.

Many figures are prepared with MAPLE and Geometer's Sketchpad. Additionally, Chapter 4 has a number of screenshots produced by a popular graphing calculator by Texas Instruments. These graphs are shown especially for the benefit of students accustomed to using calculators in order to introduce them to analytical methods. Sometimes by comparing solutions obtained numerically and analytically, we can more readily see the advantages of analytical methods while referring to the numerically calculated graphs to give us confidence in our results. Following the new rules of the US Mathematics Olympiad, I suggest that you prepare all sketches by hand and urge you not to rely on a calculator or computer to solve the homework problems.

Do I Need This Book?

You can decide how necessary this book is for you by taking the following quiz. I give problems like these in order to see how deeply students understand functions and their properties. Here they are, just three simple problems.

Find all real solutions of the following equations and inequality:

1. $\frac{1}{x} + x = \sqrt{2}$
2. $\sin(3x - 1) = 3$
3. $2^x + 2^{-x} < \sqrt[5]{5}$

Did you get the solutions? Are you still working? Is it hard?

Hint
Do not try to solve them—think about the properties of functions and use common sense.

Answer
None of the three problems has a solution. Mathematicians say that their solutions are the empty set.

Solution
Now, look at each problem again as if you are solving it from the end. Can you explain why, for example, the first equation does not have a solution? Because the right side of the equation is positive, it can have a solution only if $x > 0$. Apply to the left side of the equation the relationship between the arithmetic and geometric means of two positive numbers a and b, $a + b \geq 2\sqrt{ab}$ and observe that equality appears if and only if $a = b$. So, we see that $x + \frac{1}{x} \geq 2\sqrt{x \cdot \frac{1}{x}} = 2$. The left side is always greater than or equal to two, but the right side is a constant, $\sqrt{2}$. Because $2 > \sqrt{2}$, the first equation has no solutions for all positive x. Let us summarize our ideas:

Consider an equation of type, $x + \frac{1}{x} = c$.

1. $c > 0 \Rightarrow x > 0$, where the logical symbol "$\Rightarrow$" means "implies."
   Because $x + \frac{1}{x} \geq 2$, the equation will have a solution if and only if $c \in [2, \infty)$.
2. $c < 0 \Rightarrow x < 0$
   Because $x + \frac{1}{x} \leq -2$, the equation will have solution if and only if $c \in (-\infty, -2]$.
3. $c = 0$, then the equation has no real solutions.

We can see this analysis illustrated by example.

Example 1

$$x + \frac{1}{x} = 3, \quad 3 > 2 > 0$$

$$x^2 - 3x + 1 = 0$$

$$x_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

so that a real solution exists and both roots are positive.

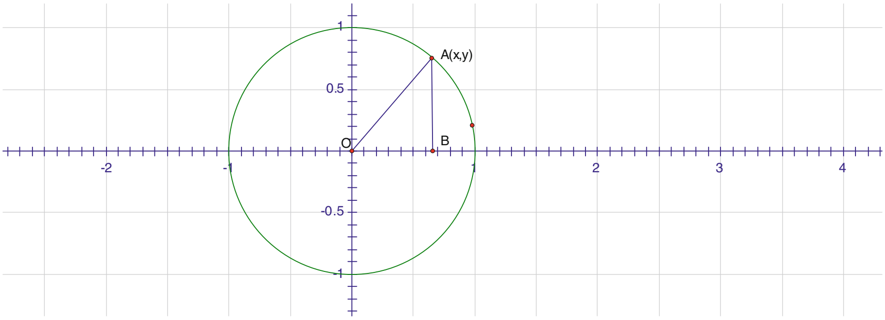


Figure P.1.1 Point on the unit circle

Example 2

$$x + \frac{1}{x} = -4, \quad -4 < -2 < 0$$
$$x^2 + 4x + 1 = 0$$
$$x_{1,2} = -2 \pm \sqrt{3}$$

so that a real solution exists and both roots are negative.

I hope that after solving this problem, the second equation will be easy to understand. We want to compare the ranges of the functions on the left and on the right. If you have completely forgotten trigonometry, let us recall the definitions of sine and cosine.

Sine is the ***y*-coordinate** of a point on the unit circle (of radius 1) and **cosine** is the ***x*-coordinate**. Note that any point on the unit circle can have coordinates only within the interval $[-1, 1]$.

Let us assume that the point (say A in Figure P.1.1) corresponds to the angle t, $\angle BOA$. Sine and cosine have the same range, $[-1, 1]$, so $-1 \le \sin t \le 1$ and $-1 \le \cos t \le 1$.

Now, can you explain why the second problem has no solution for real x?

$$\sin(3x - 1) = 3$$

The left side is always less or equal to one, but the right side is three. This equation has no solutions.

Remark
Our equation, $\sin t = 3$, will have a solution for t over the set of complex numbers.

What do you think about the third equation? It has no solutions as well. Can we prove it? The left side is positive and the right is positive. Recalling properties of exponents, we notice that 2^x and 2^{-x} are reciprocals of each other. Now we can rewrite the left side as $2^x + \frac{1}{2^x}$ so that $2^x + \frac{1}{2^x} \geq 2\sqrt{2^x \cdot \frac{1}{2^x}} = 2$. Because $2 = \sqrt[5]{32} > \sqrt[5]{5}$, the third equation has no solutions.

Notice that all of these three equations have clear graphical interpretations. Functions on the left side of the equation have no points of intersection with functions on the right.

The purpose of this book is to teach you how to apply properties of functions to solving some nonstandard problems and problems with parameters. For demonstration, I selected only problems that had no solutions. However, many problems have solutions and you need to learn how to recognize that fact and solve them. Those of you who solved all three problems without my help will find many new ideas and approaches you might enjoy.

How Should This Book Be Used?

Here are my suggestions about how to use the book. Read the corresponding section and try to solve the problem without looking at my solution. If a problem is not easy, then sometimes it is important to find an auxiliary condition that is not a part of the problem but that will help you to find a solution to the problem in a couple steps. In this book, I will show simple and challenging problems and will point out ideas we used in the auxiliary constructions so that you can develop your own experience and hopefully become an expert soon. If you find any question or section too difficult, skip it and go to another one. Later, you may come back and try to understand it. Different people respond differently to the same question. Return to difficult sections later and then solve all the problems. Read my solution when you have found your own solution or when you think you are just absolutely stuck. Think about similar problems that you would solve using the same or similar approach. Find a similar problem from the homework section. Create your own problem and write it down along with your original solution. Now it is your powerful method. You will use it when it is needed.

I promise that this book will make you successful in problem solving. If you do not understand how a problem was solved or if you feel that you do not understand my approach, please remember that there are always other ways to do the same problem. Maybe your method is better than one proposed in this book. If a problem requires knowledge of trigonometry or number theory or another field of mathematics that you have not learned yet, then skip it and do other problems that you are

able to understand and solve. This will give you a positive record of success in problem solving and will help you to attack the harder problem later. Do not ever give up! The great American inventor Thomas Edison once said, “Genius is one percent inspiration, ninety-nine percent perspiration.” Accordingly, a “genius” is often merely a talented person who has done all of his or her homework. Remember that it is never late to become an expert in any field. Archimedes himself became a mathematician only at the age of 54.

I hope that upon finishing this book you will love math and its language as I do. Good luck and my best wishes to you!

Denton, TX, USA Ellina Grigorieva

Acknowledgements

During the years working on this book I received feedback from my friends and colleagues at Lomonosov Moscow State University for which I will always be indebted.

I am especially grateful for the patient and conscientious work of Mrs. Ann Triplett and the contributions of Dr. Paul Deignan in the final formatting and preparation, and for the multitude of useful and insightful comments on its style and substance.

I appreciate the extremely helpful suggestions from the reviewers: your feedback made this book better! I would also like to acknowledge the editors Christopher Tominich and Benjamin Levitt at Birkhauser who have always been encouraging, helpful, and positive.

Denton, TX, USA Ellina Grigorieva

Contents

Chapter 1
Solving Problems Using Properties of Functions

Many students struggle to understand functions and their behavior, especially such concepts as the domain and range of a function. We very often start manipulations with an equation without thinking first about the properties of the functions. For example, the solution to the equation, $\sqrt{x} + \sqrt{x+16} = 8$, can be found immediately as $x = 9$ if, instead of the standard squaring of both sides technique, we notice that the left side of the equation is the sum of two monotonically increasing functions. This means that if a solution exists, it must be unique. We can see that $x = 9$ makes the equation true; therefore, it is the only solution.

There are many different "tricks" and properties that you will learn in this chapter, but the chapter is mainly focused on recognizing boundedness of functions and using this property in the solution of equations. Likewise, the solutions of the transcendental equation $\sin^2(\pi x) + \sqrt{x^2 + 3x + 2} = 0$ can be found if we notice that the left side is the sum of two nonnegative functions while the right side is zero. In this chapter you will learn that the solution exists only if each function on the left is zero. The purpose of this chapter is to help you understand these topics at the introductory level and demonstrate how knowledge of the properties of functions allows us to solve nonstandard problems of elementary mathematics.

1.1 Continuous Functions and Discontinuities

Let us start from the definition of a function. A **function** is a correspondence between two sets such that each element from the first set, called the **domain**, associates with only one element from the second set, called the **range**.

Functions can be given by verbal description, by a table, by ordered pairs, by a graph, by formulas, or by several formulas on different intervals for so-called piecewise functions. If you remember the definition of a function given above,

E. Grigorieva, *Methods of Solving Nonstandard Problems*,
DOI 10.1007/978-3-319-19887-3_1

then you can easily check if dependence between two sets is functional or not, no matter how that dependence is given.

Consider the relations between the sets of Figure 1.1. The first two (Figure 1.1a, b) are functions. However, the relation of Figure 1.1c is not a function because one element of the domain, X, corresponds to two elements of the range Y. The relation in Figure 1.1b is a function, $X \rightarrow Y$ (the single arrow "→" means that there is a

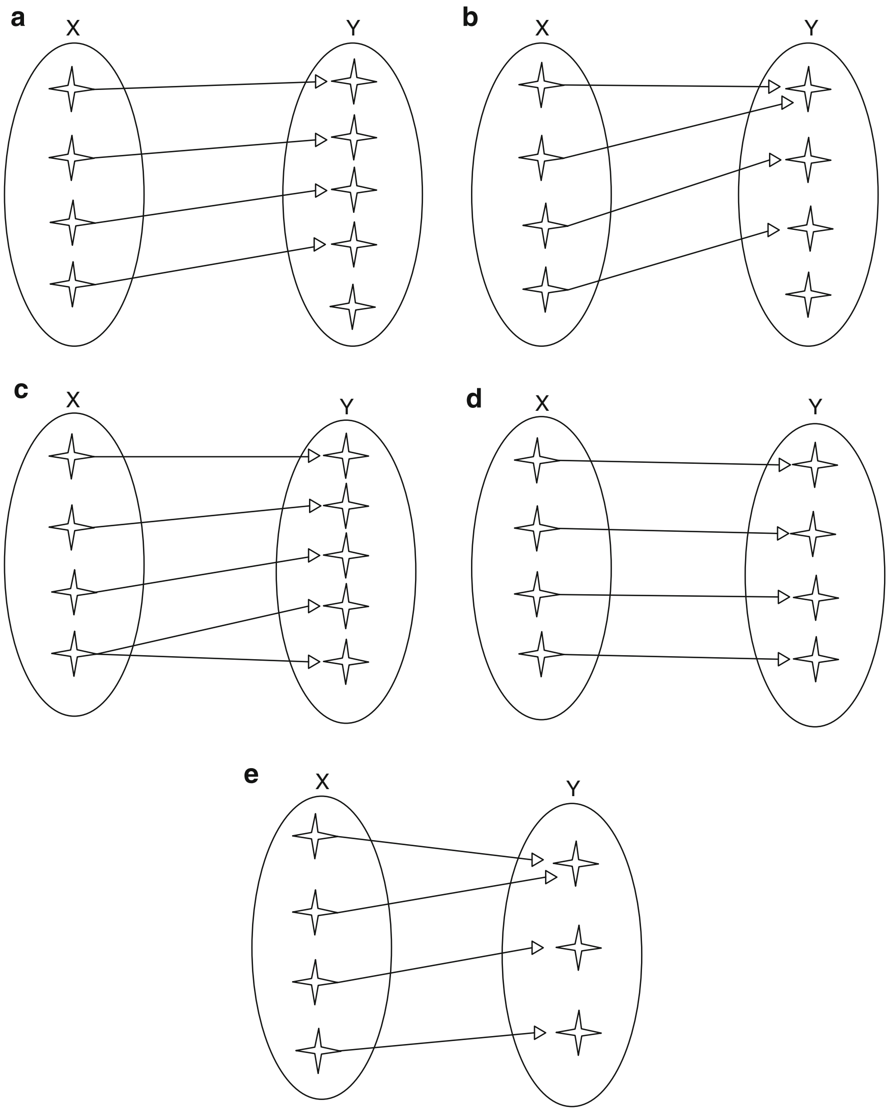


Figure 1.1 (**a**) Each passenger takes one seat. (**b**) Mother and baby take one seat. (**c**) One passenger takes two seats. (**d**) One-to-one function. (**e**) Onto

functional relation between the sets), because each element of X associates with only one element of Y. It is allowed that more than one element of the domain, X, corresponds to same element of Y. In order to help you to remember these cases, I would like to offer you the following verbal memorable example.

Assume that you are flying from DFW to Costa Rica by American Airlines during the rainy season. What would you expect? Probably some seats will not be occupied. Suppose that the set X is the set of all passengers of the plane and set Y is the set of all seats. If each passenger takes only one seat then such a relationship is a function even though some seats are not occupied. This is the case of Figure 1.1a. Next, consider that some passengers have babies and hold their babies on their laps, not on a separate seat. Then the relation is still a function because the baby has only one seat just as the parent does. This is the case of Figure 1.1b. Finally, if there were empty seats and some passengers decided to take two or three seats, then such relationship is not a function (Figure 1.1c). Moreover, if all seats are taken and each passenger occupies only one seat then the relation between the two sets is a **one-to-one** function (Figure 1.1d). In the case when the flight is full and all seats are occupied but babies sit on the laps of their parents, then such a relationship is not one to one but **onto** (Figure 1.1e).

If you have this story in mind, you will be able to find out if a given relation between two variables is a function or not. For example, assume that three relations are given by their graphs in the plane (Figure 1.2). Note that a function may be given by the formula, "all seats are always occupied," so we can talk about relations through graphs. To check if a relation is a function, you can use the **vertical line test**. Mentally draw several vertical lines that go through the graph. If a line intersects the graph at most at one point, then it is a function. Further if each horizontal line crosses each function at most at one point, then such a function is one to one.

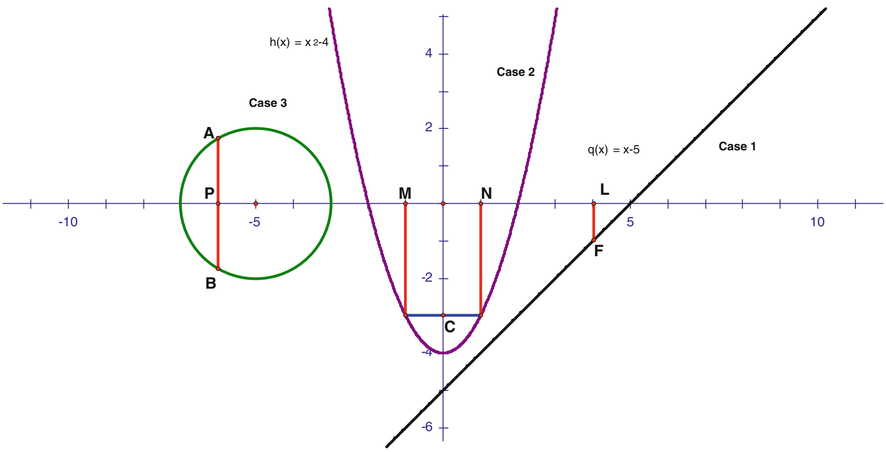


Figure 1.2 Relation vs. function

Case 1 We have a one-to-one function (any linear function is always one to one) and the relation is one element of the range for each element of the domain (i.e., passenger L takes only one seat F and vice versa).

Case 2 A parabola is not a one-to-one function. It is like the case when the mother keeps her baby on her lap (here “mother” M and “baby” N take the same seat C).

Case 3 A circle is not a function! One value of X associates with two different values of Y (passenger P takes both seats A and B).

Usually we deal with functions given by formulas in the form of $y = f(x)$. Denote the domain of a function by $D(f)$ and the range by $R(f)$. A function f assigns a numeric value $f(x)$ to each point X in the domain $D_f = D(f)$ on which f is defined. The range of f is the set $R_f = R(f) = \{f(x)|x \in D(f)\}$. For example,

1. $y = f(x) = x^3 - 3x^2 - 7$,
2. $y = f(x) = 3\sin x + 4\cos x$,
3. $y = f(x) = \frac{1}{x-2} + \frac{1}{x+2}$, or
4. $y = f(x) = \sqrt{x-3}$ are examples of functions.

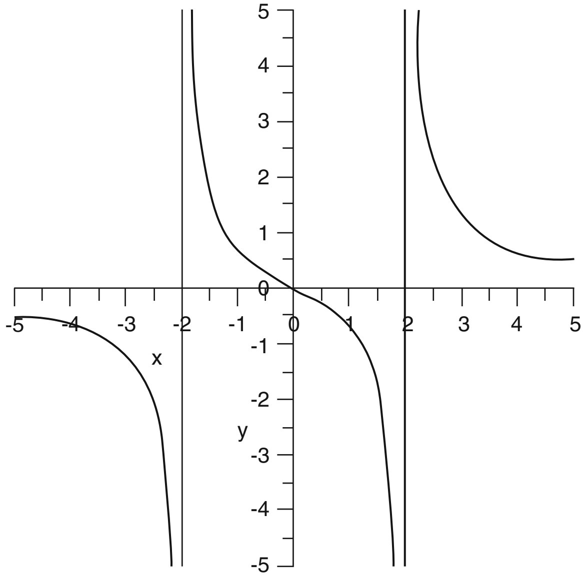
In order to find the domain, you need to ask a question, “Are there any restrictions on the independent variable, x?” That is, “Can I replace x by any number?”

The first function is polynomial and all polynomial functions are defined on the entire real number line so that you can replace x by any real number, i.e., $D(f): x \in (-\infty,\infty)$ where the symbol “$\in$” is set notation for “is an element of.” The second function is a combination of sine and cosine functions and so it also is defined on the entire number line. The third function consists of two rational functions (fractions) and we know that division by zero is not allowed. Therefore, for any function that has the form $f(x) = \frac{M(x)}{N(x)}$ we would exclude all values of the independent variable that make the denominator zero.

If a function is defined for all real numbers such that the dependent variable of the function varies smoothly (if at all) with changes in the independent variable, then we say that the function is everywhere **continuous**. If a function has a restriction on x, then it has a discontinuity at those values of x. For example, while the first and the second functions are defined and continuous for all real x, the third function has a discontinuity at $x = 2$ and at $x = -2$. The graph of $y = f(x) = \frac{1}{x-2} + \frac{1}{x+2}$ will have interruptions at the discontinuity points. If you want to sketch a graph of a discontinuous function, you cannot do it without lifting a pencil from the piece of paper on which you do your sketch. For the third function, you would need to lift it at both discontinuity points.

The domain is the union of three intervals on which our function is continuous, $D(f): x \in (-\infty,-2)\cup(-2,2)\cup(2,\infty)$. Readers who can find the domain analytically would not be surprised by the look of the function produced by many graphing calculators. For example, they would know that the vertical lines $x = 2$ and $x = -2$ shown in Figure 1.3 are the vertical asymptotes and not a part of the graph.

Figure 1.3 A function with two vertical asymptotes



The last function, $y = f(x) = \sqrt{x-3}$ (Figure 1.4), is a radical of even degree so that the domain of the function is restricted to the nonnegative radicand values that can be written as the inequality $x - 3 \geq 0$ or $x \geq 3$. There will be no graph for any value of the independent variable less than three, i.e., $D(f) : x \in [3,\infty)$.

Each component of a more complicated function must be examined to determine the restrictions on the domain of the composite function. The domain of the composite function will be the intersections of the restrictions of the domains of the components. Let us do the following problem:

Problem 1 Find the domains of $f(x) = \dfrac{\sqrt{x-1}}{x-4}$ and $g(x) = \dfrac{x-4}{\sqrt{x-1}}$.

Solution The functions are reciprocals of each other and have some similarities in the restrictions but they have absolutely different domains. The first function looks like a fraction and so we need to exclude all values of x that make its denominator zero, i.e., $x \neq 4$. Furthermore, the numerator is a square root function and so it must be defined only for nonnegative radicand values, i.e., $x - 1 \geq 0$. Solve all restrictions as a system of equations: $\begin{cases} x-4 \neq 0 \\ x-1 \geq 0 \end{cases} \Leftrightarrow D(f) : x \in [1,4) \cup (4,\infty)$ where the symbol "$\Leftrightarrow$" means "if and only if" or "equivalently." Then $x = 4$ is a vertical asymptote where the function $f(x) = \dfrac{\sqrt{x-1}}{x-4}$ has a discontinuity in its graph.

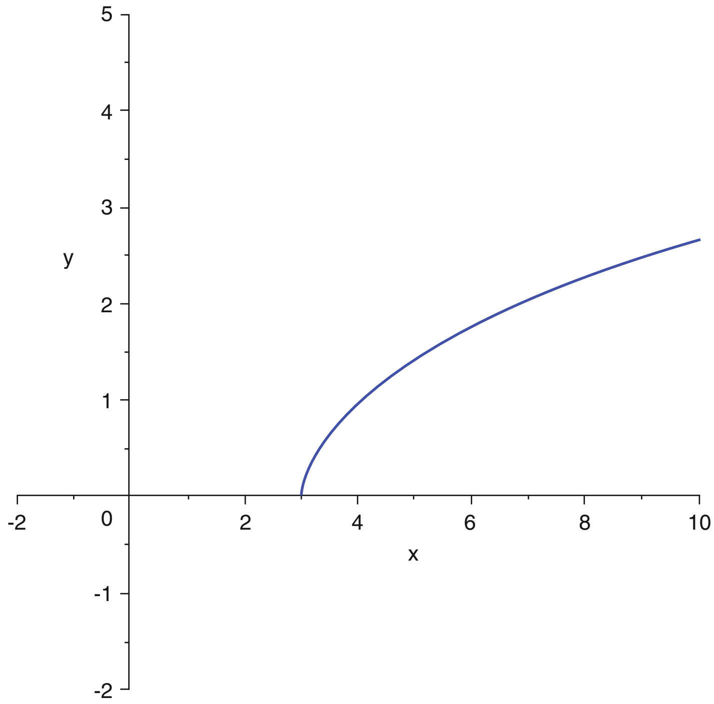


Figure 1.4 $f(x) = \sqrt{x-3}$

The second function has restrictions coming from the denominator only: its denominator cannot be zero and the radicand must be nonnegative. This can be written as $\begin{cases} x-1 \neq 0 \\ x-1 \geq 0 \end{cases} \Leftrightarrow D(g) : x \in (1, \infty)$. The second function has no vertical asymptote. The two functions are shown together in Figure 1.5.

Next, we will show how to find the range of a function. Usually, it is not as easy as finding the domain, but it is important in certain cases. Right now, I will give you some ideas on how to find the range of some functions analytically if a function is given as a formula. You will learn much more about the subject in the section focusing on bounded and unbounded functions.

Let us again consider the four functions given at the beginning of this section. Polynomial functions of odd degree can take any values for x and hence they can take any values for $y = f(x)$ from negative infinity to positive infinity. Therefore, the range of f can be written as $R(f) : y \in (-\infty, \infty)$. Polynomial functions of even degree are bounded from either above or below and cannot take any possible value.

Using an auxiliary argument, the second function can be written as $y = f(x) = 3\sin x + 4\cos x = 5\sin(\varphi + x)$ where $\varphi = \arccos\frac{3}{5}$. The range of $f(x)$ is $y \in [-5,5]$ because sine is a bounded function. We can write this as $R(f) : y \in [-5,5]$. For a better understanding of this topic and the formula's derivation, please see Chapter 3.

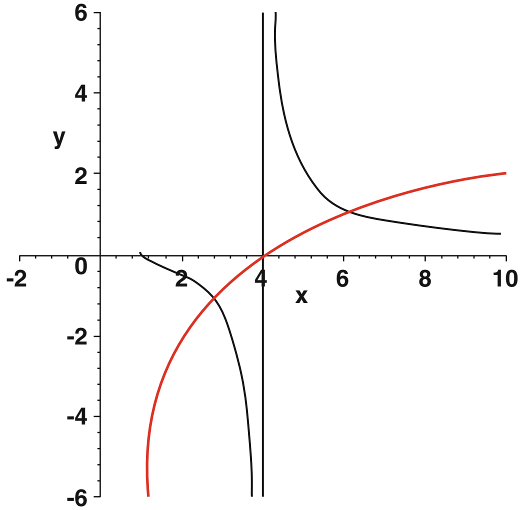


Figure 1.5 Sketch for Problem 1

If we look at the graph of function $y = f(x) = \frac{1}{x-2} + \frac{1}{x+2}$, then we can state that the variable y takes on all possible values from minus infinity to plus infinity including zero. So its range is the same as for the first polynomial function, $R(f) : y \in (-\infty,\infty)$. However, this is not typical for a rational function and each case must be investigated separately. For example, if we instead take a function $y = f(x) = \frac{1}{x-2}$, the numerator is one and never becomes zero, so the function itself can never have zero as its value—zero is not in its range. The range is $R(f) : y \in (-\infty,0) \cup (0,\infty)$.

The square root function, $y = f(x) = \sqrt{x-3}$, is never negative, so $R(f) : y \in [0,\infty)$ and takes the value $y = 0$ at $x = 3$.

We say that a function $y = f(x)$ is continuous on $[a,b]$ if it is continuous at every point of this interval. If it is continuous, it takes on all values between $f(a)$ and $f(b)$ over the segment, $[a,b]$. This is an important result, which is commonly stated as a theorem:

Intermediate Value Theorem (IVT) If $f(x)$ is a continuous function on $[a,b]$, then for every d between $f(a)$ and $f(b)$, there exists at least one value c in between a and b such that $f(c) = d$.

Corollary *If $f(x)$ is a continuous function on $[a,b]$ such that $f(a)$ and $f(b)$ are opposite in sign, then there exists at least one zero of the function between a and b such that $f(c) = 0$.*

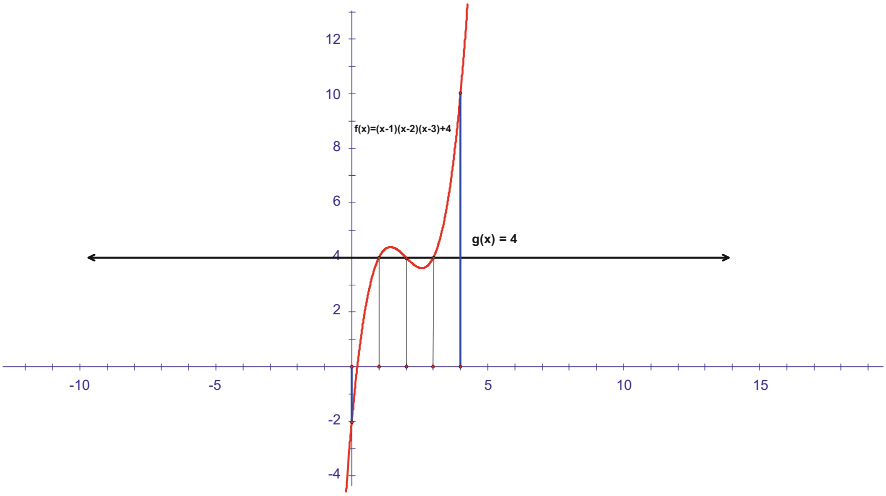


Figure 1.6 Sketch for Problem 2

Let us solve the following problem:

Problem 2 Give an example of a function that satisfies the IVT and takes a value of 4 at $x = 1$, $x = 2$, and $x = 3$.

Solution This problem does not have a unique solution.
For example, a polynomial function $f(x) = (x - 1)(x - 2)(x - 3) + 4$, shown in Figure 1.6, is continuous everywhere on the real number line including the segment $[0,4]$ where $f(0) = -2$ and $f(4) = 10$. By the IVT for each $d \in [-2,10]$ there must be at least one number $c \in [0,4]$, such that $f(c) = d$. If we choose $d = 4$, then there are exactly three values, $c_1 = 1$, $c_2 = 2$, and $c_3 = 3$ such that $f(1) = f(2) = f(3) = 4$. The graph of $g(x) = 4$ intersects $f(x)$ at three points. You can see that for other values of $d \in [-2,10]$, for example, $d \in \{2,6\}$, there exists only one point c. Also notice that the lines tangent at local minimum or maximum of the function intersect the graph at two points. Finally, the function can be written as $f(x) = x^3 - 6x^2 + 11x - 2$.

Also, there is the non-polynomial function, $h(x) = \frac{(x-1)(x-2)(x-3)}{\sqrt{25-x^2}} + 4$. This function is defined for all $x \in D(h) = (-5,5)$. Therefore it is defined on $[0,4]$ where $h(0) = 2.8$ and $h(4) = 6$. Thus $d = 4$ and the graph of $g(x) = 4$ intersects $h(x)$ at three requested points.

Answer $f(x) = x^3 - 6x^2 + 11x - 2$ or $h(x) = \frac{(x-1)(x-2)(x-3)}{\sqrt{25-x^2}} + 4$.

Let $y = f(x)$ be a function defined on $[a,b]$ and let x_1 and x_2 be any two points in $[a,b]$. Then we can say the following:

a. $f(x)$ increases on $[a,b]$ if for any $x_2 > x_1$, where $x_1, x_2 \in [a,b]$, then $f(x_2) > f(x_1)$.
b. $f(x)$ decreases on $[a,b]$ if for any $x_2 > x_1$, where $x_1, x_2 \in [a,b]$, then $f(x_2) < f(x_1)$.

A function is **monotonic** function if its first derivative is continuous and does not change sign. Monotonic function is a function which is either entirely nonincreasing or nondecreasing.

If $f(x)$ is continuous on $[a,b]$ and $f(a)$ is opposite in sign of $f(b)$, then there is at least one point c between a and b such that $f(c) = 0$. It follows that a continuously increasing or decreasing function has at most one real zero on $[a, b]$. We can rephrase it as follows:

Theorem 1 *If the function $y = f(x)$ is continuously increasing or decreasing in set I, then the equation $f(x) = 0$ has at most one solution in I.*

This is a very important statement and it means that if $f(x)$ satisfies Theorem 1 and $f(x) = 0$ has a solution, then this is the only solution of the equation.

Theorem 2 *If the function $y = f(x)$ is increasing (decreasing) over interval I, then the function $y = -f(x)$ is decreasing (increasing) on I.*

Theorem 3 *The sum of two increasing functions is an increasing function and the sum of two decreasing functions is a decreasing function.*

Please prove this statement as homework problem 9 after this chapter.

Corollary *If one of the two functions f and g is increasing and the other is a decreasing function of real variable x, then the equation $f(x) = g(x)$ has at most one real zero.*

Proof Rewrite the equation as $h(x) = f(x) - g(x) = f(x) + (-g(x)) = 0$.

Without loss of generality, we can assume that $f(x)$ is monotonically increasing and $g(x)$ is decreasing. Then the function $-g(x)$ is monotonically increasing. Next, the function $h(x)$ is the sum of two monotonically increasing functions and therefore can have at most one zero.

Note that the sum of increasing and decreasing functions is sometimes not even monotonic.

Problem 3 Solve the equation $\sqrt{2x-3} + \sqrt{4x+1} = 4$.

Solution By guessing and checking we find that $x = 2$ is a solution of the equation. Let us prove that there are no other solutions. Regarding Theorem 3, the function $f(x) = \sqrt{2x-3} + \sqrt{4x+1} - 4$ is increasing as it is the sum of two increasing functions over the entire domain $x \geq 1.5$. If it has a zero at $x = 2$, then by Theorem 1, there are no other roots.

Answer $x = 2$.

Note: You can also solve this problem the long way by squaring both sides, simplifying, and then selecting roots satisfying the domains of both square root functions.

Theorem 4 *The product of increasing (decreasing) positive functions is also an increasing (decreasing) function.*

Proof If $f(x) > 0$, $g(x) > 0$ and both are increasing functions, then $\forall x_2 > x_1$, $f(x_2)g(x_2) - f(x_1)g(x_1) = [f(x_2) - f(x_1)]g(x_2) + [g(x_2) - g(x_1)]f(x_1) > 0$.

Because each expression inside brackets is positive, then the inequality will also be true if one of the given functions is positive and the other is nonnegative.

Theorem 5 *The composition of two increasing or both decreasing functions is an increasing function.*

Proof Assume that $h(x) = (f \circ g)(x) = f(g(x))$ and let $x_1, x_2 \in D(h)$ and $x_2 > x_1$. There are two possible cases:

Case 1 If $f(x)$ and $g(x)$ are both increasing functions, then $g(x_2) > g(x_1)$ and $f(g(x_2)) > f(g(x_1))$. Hence, the function $h(x)$ is an increasing function.

Case 2 If $f(x)$ and $g(x)$ are both decreasing functions, then $g(x_1) > g(x_2)$. Because $f(x)$ is a decreasing function and $g(x_1) > g(x_2)$, then $f(g(x_1)) < f(g(x_2))$, and then $h(x_2) > h(x_1)$. Therefore $h(x) = (f \circ g)(x)$ is again an increasing function.

Hence, the function $h(x)$ is an increasing function.

Example Assume that we want to discuss whether the function $h(x) = \sqrt{x^3 + 2x - 3}$ is increasing or decreasing on the domain. We can look at this function as the composition of two other functions as $h(x) = (f \circ g)(x)$, where $f(x) = \sqrt{x}$ and $g(x) = x^3 + 2x - 3$. Because both functions are monotonically increasing, their composition, $h(x)$, is also monotonically increasing.

On the other hand, if $f(x) = -2x + 1$ and $g(x) = -x^3$ (both decreasing functions), both compositions $h(x) = (f \circ g)(x) = 2x^3 + 1$ and $w(x) = (g \circ f)(x) = 8x^3 - 12x^2 + 6x - 1$ are increasing functions.

Theorem 6 *The composition of a decreasing and an increasing function or an increasing and a decreasing function is a decreasing function.*

The proof of this statement is given as a Homework problem 14 (after this chapter).

Theorem 7 *If the function $f(x)$ is monotonic on the set, X, and keeps the same sign over X, then the function $g(x) = \frac{1}{f(x)}$ has opposite monotonic behavior on X.*

Proof Let $x_2 > x_1$ where $\{x_1, x_2\} \in X$, and consider the difference $g(x_2) - g(x_1)$ and transform it as

$$\begin{aligned} g(x_2) - g(x_1) &= \frac{1}{f(x_2)} - \frac{1}{f(x_1)} \\ &= \frac{f(x_1) - f(x_2)}{f(x_2) \cdot f(x_1)} \end{aligned}$$

1. Because the function $f(x)$ keeps the same sign in X, then the product inside the fraction's denominator is always positive.
2. The difference in the numerator is either positive if $f(x)$ is monotonically decreasing or negative if $f(x)$ is increasing on X.

Therefore $g(x)$ is decreasing if $f(x)$ is increasing and vice versa. The statement is proven.

Problem 4 Investigate if the function $f(x) = \frac{1}{x-2} + \frac{1}{x+2}$ is increasing or decreasing and give its intervals of monotonic behavior.

Solution The function is the sum of two decreasing functions, so it must decrease over the entire domain, $D(f) : x \in (-\infty,-2) \cup (-2,2) \cup (2,\infty)$.

Problem 5 Solve the equation $\left(\sqrt{2-\sqrt{3}}\right)^x + \left(\sqrt{2+\sqrt{3}}\right)^x = 2^x$.

Solution Let us solve this problem by applying properties of monotonically decreasing functions. Both sides are positive. Dividing both sides by 2^x, we obtain

$\left(\frac{\sqrt{2-\sqrt{3}}}{2}\right)^x + \left(\frac{\sqrt{2+\sqrt{3}}}{2}\right)^x = 1$. It is not hard to see that the left side of the equation is the sum of two decreasing exponential functions because their bases are $\frac{\sqrt{2-\sqrt{3}}}{2} < 1$ and $\frac{\sqrt{2+\sqrt{3}}}{2} < \frac{\sqrt{2+\sqrt{4}}}{2} = \frac{2}{2} = 1$ which is less than one. Then the function on the left is decreasing and takes each value once, and the equation has only one solution. We can see that $x = 2$ satisfies the equation because $\frac{2-\sqrt{3}}{4} + \frac{2+\sqrt{3}}{4} = 1$.

Answer $x = 2$ and it is unique.

Problem 6 How many X-intercepts does the function $f(x) = x^5 + x^3 + 1$ have?

Solution We can look at this function as the sum of two monotonically increasing functions, x^5 and $x^3 + 1$, so that $f(x)$ can have only one x-intercept that is the real root of the equation $f(x) = 0$. Because $f(-1) = -1 < 0$, $f(0) = 1 > 0$ then the root belongs to the interval $x_0 \in (-1,0)$. Additionally, if you want to use a derivative, then you can find that $f'(x) = 5x^4 + 3x^2 > 0$ and this proves that the function is monotonically increasing and will take each value only once, including the value of zero.

1.2 Bounded and Unbounded Functions

A function $f(x)$ is **bounded below** if for any $x \in D(f)$, $f(x) \geq a$ where a is some real number. For example, $f(x) = \sqrt{x} - 2$ is bounded below and lies everywhere above the line $y = -2$. The function is monotonically increasing on its domain $x \in D(f) = [0,\infty)$. A function $f(x)$ is **bounded above** if for any $x \in D(f)$, $f(x) \leq b$, where b is some real number. For example, $f(x) = 5 - x^2$ is bounded above, and 5 is its maximum value, so for every x, $f(x) \leq 5$.

A function $f(x)$ is called a **bounded function** if there exist two real numbers a and b such that if for any $x \in D(f)$, $a \leq f(x) \leq b$. For example, the sine and cosine functions are bounded since $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$. The function $f(x) = \frac{5}{2+3\sin^2 x}$ is also bounded such that $1 \leq f(x) \leq 2.5$. Also, the function $g(x) = 2 + 3\sin^2 x$, such that $2 \leq g(x) \leq 5$ (both are shown in Figure 1.7).

The function $f(x) = \frac{5}{2+3x^2}$ is also bounded such that $0 < f(x) \leq 2.5$. Obviously, because the numerator is five the function never becomes zero but only approaches the x-axis as x goes to plus or minus infinity (Figure 1.8).

Let us practice solving problems.

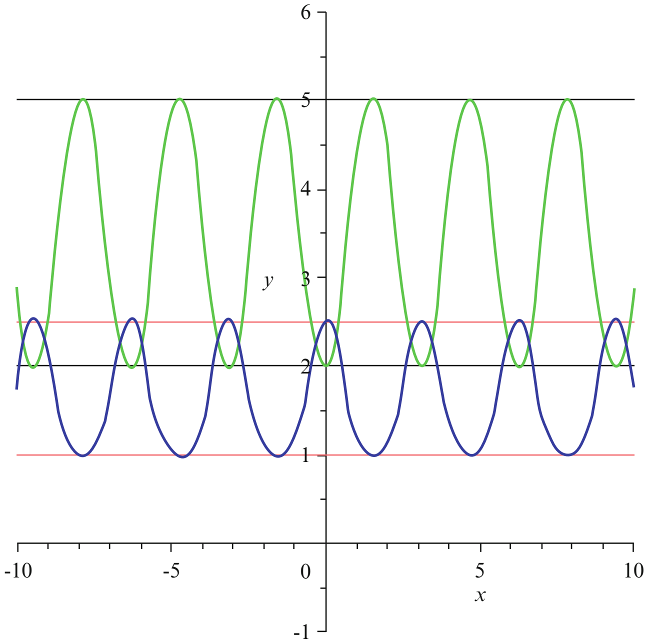
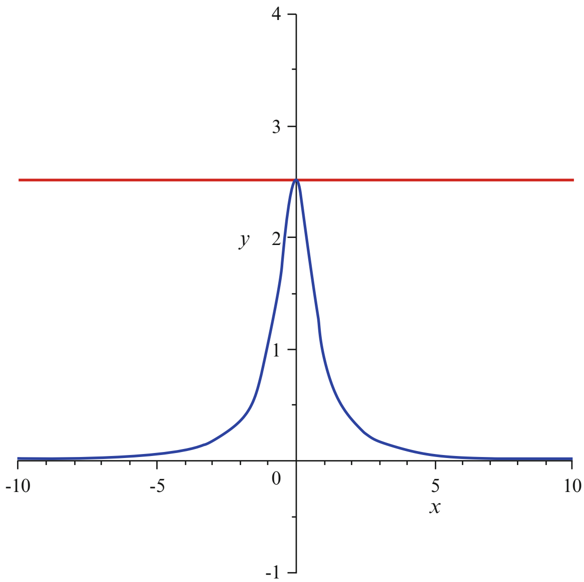


Figure 1.7 Two bounded functions

Figure 1.8 $f(x) = \frac{5}{2+3x^2}$



Problem 7 Is the function $f(x) = \sqrt{4 - |x|}$ a bounded or unbounded function?

Solution Let b be a possible value of the function $f(x)$. Let us find the solution to the equation $\sqrt{4 - |x|} = b$ where $b \geq 0$. Squaring both sides, we obtain $4 - |x| = b^2$ or $|x| = 4 - b^2$. Since $|x| \geq 0$, the right side must also be greater than or equal to zero, i.e., $4 - b^2 \geq 0$. Solving this, we obtain possible values of b:

$$\begin{cases} |b| \leq 2 \\ b \geq 0 \end{cases} \Leftrightarrow 0 \leq b \leq 2.$$

Therefore the function is bounded, $0 \leq f(x) \leq 2$.

Answer $R(f) : y \in [0, 2]$.

Problem 8 Is the function $f(x) = \frac{5}{2+3\sin x}$ bounded?

Solution Let us find the domain of this function first. Because we cannot divide by zero, the domain of this function is restricted such that $\sin x \neq -\frac{2}{3}$. $D(f) : x \neq (-1)^n \arcsin\left(-\frac{2}{3}\right) + n\pi$ where $n = 0, \pm 1, \pm 2, \ldots$. The vertical lines in the graph of Figure 1.9 are the vertical asymptotes. In order to answer the question about boundedness, we could again consider $\frac{5}{2+3\sin x} = b$, but instead, we will consider only the denominator of this fraction: $g(x) = 2 + 3\sin x$ where $-1 \leq \sin x \leq 1$, so $-1 \leq g(x) \leq 5$.

With $f(x) = \frac{5}{g(x)}$ we have $f(x) \leq -5$ and $f(x) \geq 1$, so $R(f) = (-\infty, -5] \cup [1, \infty)$ as shown in Figure 1.9.

Answer The function is unbounded, but does not take values between -5 and 1.

Problem 9 Solve the equation $x^3 + x + 10 = 0$ over the set of real numbers.

Solution It is easy to find that $x = -2$ is the root of the equation. Let us prove that it is the only root. Consider two functions: $f_1(x) = x^3$ and $f_2(x) = x + 10$. Both functions are strictly increasing on the entire real number line, so $f(x) = f_1(x) + f_2(x)$ is also strictly increasing on R. Therefore, $f(x)$ is one to one. In particular, $f(x) = 0$ only at $x = -2$.

Answer $x = -2$.

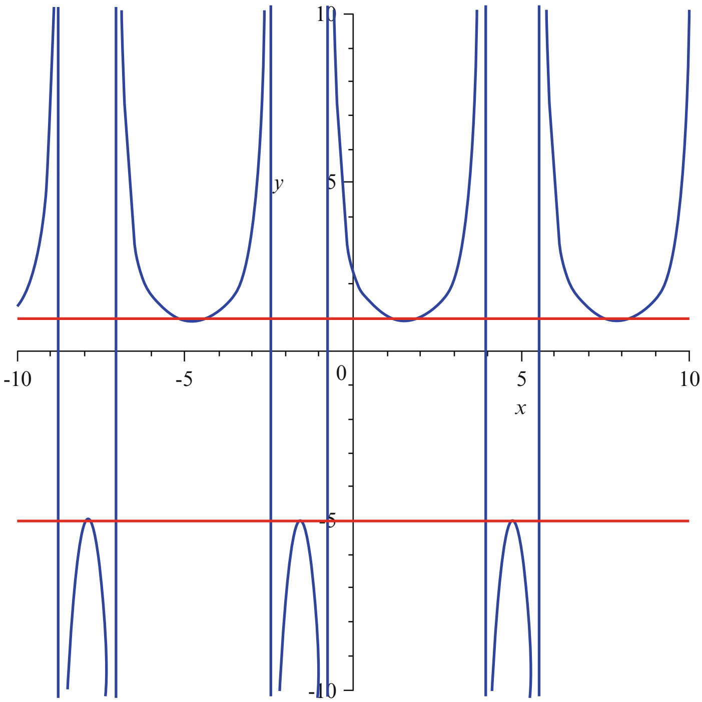


Figure 1.9 Sketch for Problem 8

1.3 Maximum and Minimum of a Function

A function has a **relative minimum** at $x = c$ if the function evaluated at $x = c$ is less than at any other point in the neighborhood of $x = c$. A relative minimum is the lowest point in an open interval, but not necessarily over the entire domain. The interval of the neighborhood can be infinitesimally small. If there is no lesser valued point on the domain of the function, then the relative minimum would also be a **global minimum**, or simply, **minimum** of the function.

A function has a **relative maximum** at $x = c$ if the function evaluated at $x = c$ is greater than at any other point in the neighborhood of c no matter how small that neighborhood may be. A relative maximum is the greatest valued point of a function in an open interval, but not necessarily over the entire domain. Likewise, if there is no point throughout the entirety of the domain for which the function takes a greater value, then that point would be a **global maximum**, or simply, the **maximum** of the function.

Usually if a function is differentiable, then its local extrema can be found using the first derivative test. For a differentiable function, $f(x)$, its local maximum and minimum satisfy the zero of its first derivative. The function is increasing on $x \in (a, b)$ if its first derivative keeps a positive sign on the entire interval. Likewise,

the function $f(x)$ is decreasing on $x \in (a, b)$ if its first derivative is negative along the entire interval. If a function f is monotonically increasing (decreasing) on (a, b) and is defined on $[a, b]$, then $f(a)$ is the global minimum (maximum) and $f(b)$ is the global maximum (minimum).

There are many theorems in a calculus or mathematical analysis course that pertain to functions. Two of the most important are Rolle's Theorem and Lagrange's Mean Value Theorem.

Theorem 8 (Rolle's Theorem) *Let f be a function that is continuous on* $[a,b]$ *and differentiable on* (a,b) *and for which* $f(a) = f(b) = 0$. *There exists at least one point* $c \in (a, b)$ *for which* $f'(c) = 0$.

Theorem 9 (Lagrange's Mean Value Theorem) *Let f be a function that is continuous on [a,b] and differentiable on (a,b). There exists at least one point* $c \in (a, b)$ *for which* $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Let us demonstrate the importance of Rolle's Theorem. If we did not see the graph or even the formula of the function $f(x)$, but did know that it is continuous and differentiable for all real numbers and that $f(-1) = f(1) = f(3) = 0$, then we could state using Rolle's Theorem that there must be at least one local extremum on $x \in (-1, 1)$ and additionally at least one on $x \in (1, 3)$. For example, the function $f(x) = (x - 1)(x + 1)(x - 3) = x^3 - 3x^2 - x + 3$ is defined for all real numbers and it is unbounded. However, we can see that it has the two special points: A (relative maximum) and B (relative minimum) (Figure 1.10).

We can find its extrema and intervals of increasing and decreasing by the first derivative test and its inflection point using the second derivative:

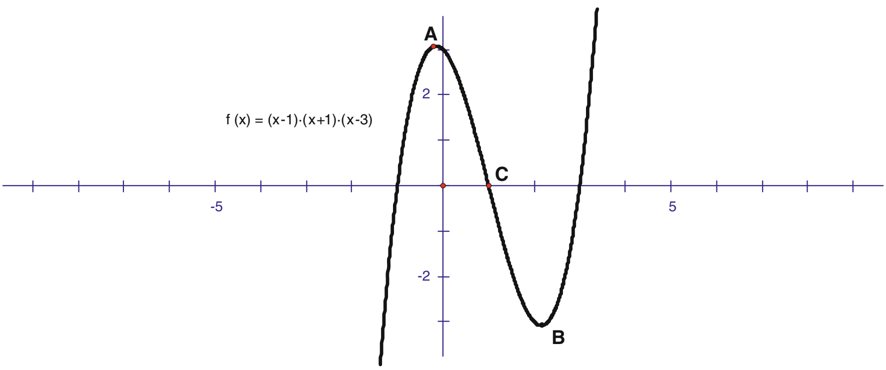


Figure 1.10 Illustration of Rolle's Theorem

$$
\begin{aligned}
f^{'}(x) &= 3x^2 - 6x - 1 = 0 \\
x &= \frac{3 + 2\sqrt{3}}{3} \qquad \text{(local min, } B\text{)} \\
x &= \frac{3 - 2\sqrt{3}}{3} \qquad \text{(local max, } A\text{)} \\
f^{''}(x) &= 6x - 6 = 0
\end{aligned}
$$

Even without the formula, we could predict at least two extrema for the functions that both fit the condition of the problem. Of course, $f(x) = (x-1)(x+1)(x-3)$ is just one of such possible functions.

Problem 10 Check the validity of Rolle's Theorem for the function $f(x) = x^3 + 3x^2 - x - 3$.

Solution Function $f(x)$ is continuous and differentiable for all values of x. Moreover, $f(x) = (x+3)(x+1)(x-1)$. Since $f(x)$ is continuous on $[-3, 1]$ and $f(-3) = f(-1) = 0$, then by Rolle's Theorem,[1] $\exists \xi_1 \in (-3, -1)$ s.t. $f^{'}(\xi_1) = 0$. Since $f(x)$ is continuous on $[-1,1]$ and $f(-1) = f(1) = 0$, then $\exists \xi_2 \in (-1, 1)$ s.t. $f^{'}(\xi_2) = 0$. The zeroes of the derivative of f are found

$$
\begin{aligned}
f^{'}(x) &= 3x^2 + 6x - 1 = 0 \\
\xi_{1,2} &= \frac{-3 \mp 2\sqrt{3}}{3}
\end{aligned}
$$

where $-3 < \xi_1 < -1$ and $-1 < \xi_2 < 1$.

Problem 11 The function $f(x) = 1 - \sqrt[3]{x^2}$ has zeroes at $x = -1$ and $x = 1$. However, $f^{'}(x) \neq 0 \; \forall x \in (-1, 1)$. Explain what seems to be a contradiction with Rolle's Theorem.

Solution Given that $f(x) = 1 - \sqrt[3]{x^2}$, then $f^{'}(x) = \frac{-2}{3\sqrt[3]{x}}$. However, $f'(x)$ is undefined on $(-1,1)$, i.e., at $x = 0$. Therefore, there is no contradiction. Rolle's Theorem does not apply because f is not differentiable at zero, so it does not meet the condition of being differentiable on the entire open interval $(-1,1)$.

The importance of Lagrange's Theorem also cannot be overlooked: this theorem is very important in the study of functions' properties and their graphs. Many of you remember from calculus that if on a given closed interval the sign of the first

[1] When written as part of a mathematical predicate, "there exists" is often abbreviated as, $\exists$ and "for all" is written as $\forall$. Likewise, we will often abbreviate "such that" as "s.t.". The purpose of these abbreviations is to make the mathematical ideas structurally compact as we extend these concepts out to become building blocks for more complicated mental structures.

derivative of a function is positive, then the function is increasing on this interval, and if the sign of the derivative is negative on some closed interval then the function is decreasing on that interval. Obviously the function must be defined on the corresponding closed intervals. The proof of this fact can be done with the use of Lagrange's Theorem as follows. Assume that the function $f(x)$ is defined on $[a,b]$ and has a positive derivative at each point of this interval, other than possibly at the ends of the interval; then the function is increasing on $[a,b]$.

More precisely, consider two values of the independent variable, $a_2 > a_1$, a_1, $a_2 \in [a, b]$; then for an increasing function, the following must be true: $f(a_2) > f(a_1)$.

Thus, it follows from Lagrange's Theorem that

$$f(a_2) - f(a_2) = f^{'}(c)(a_2 - a_1)$$

where c is an internal point of the interval, $c \in [a_1, a_2]$; hence c is also an internal point of the interval $[a,b]$. If the derivative is positive, then the right-hand side of the equation above is positive and our statement is proven.

As a real-life application of Lagrange's Theorem, I can tell you a story about a speeding ticket. Suppose that you travel from home to school and usually accelerate from time to time because you do not want to be late for class. Unfortunately, you get a ticket this time; a policeman stopped you and the ticket states that you were going 5 mi over the speed limit. You are not happy because you really slowed down when you saw a police car … to 50 mi/h, within the speed limit of 60 mi/h. Lagrange's Theorem can explain why it is useless to argue with any policeman on the subject of speeding. Imagine your travel distance as a function of time, so that for any selected time, the instantaneous velocity (speed) is the value of the slope of the tangent line at that point. If you do not change your speed, then the distance curve is a piece of a straight line and the average speed equals your actual speed. If your speed is changing over time, for example, as a curve similar to one shown in Figure 1.4, then at each point a velocity is different. If you mentally connect the end points (at $x = 3$ and $x = 10$) then the slope of the secant line $\frac{\sqrt{7}}{7}$ is the average speed. You can notice that a tangent line near the starting point of the curve will have a large value for the slope (your "speed" is higher there than the average), and at the points close to the end of the given interval, the slope is smaller than the "average" (you started to slow down). You also can see that there is a point on the curve at which the slope of the tangent line equals to the slope of the secant line. The policeman calculated your average speed between point A and point B of the actual road by dividing the total distance between the two points by the total traveling time and obtained 65 mi/h, that is, 5 mi more than the speed limit, 60. If your average speed on segment AB was 65, then by Lagrange's Theorem, there was at least one moment of time t at which your instantaneous velocity was 65 mi/h, and even though at the last moment you slowed down to 50 mi/h, it is fair to give you a speeding ticket ….

1.4 Even and Odd Functions

In order to solve nonstandard problems we need to recognize functions with special properties. Let us consider the following problem:

> Find all solutions of the equation: $x^6 + 2x^4 + 3x^2 = 6$.

Properties of polynomial functions and different methods of solving them are well explained in Chapter 2. However, by guessing and checking we can see that $x=1$ is the solution. Because on the left we have only even powers of x then if $x=a$ is the solution to this equation then $-a$ is also a solution. You can check that $x=-1$ is also a solution. There are many functions with similar properties such as $f(x) = x^6 + 2x^4 + 3x^2$. They are called even functions. The function $f(x)$ is an **even** function if for any $x \in D(f)$, $f(-x) = f(x)$. What can we additionally say about all even functions?

1. $a \in D(f) \Rightarrow -a \in D(f)$ where the symbol "$\Rightarrow$" means "implies."
2. The graph of an even function is symmetric with respect to the y-axis.

For example, $f(x) = \frac{5}{2+3x^2}$ is an even function, as is $f(x) = \sqrt{4-|x|}$. If you ever doubt if a function is even or not, then evaluate $f(-x)$ and compare it with $f(x)$; they must be the same.

The next class of special functions is odd functions. The function $f(x)$ is an **odd** function if for any $x \in D(f), f(-x) = -f(x)$. What can we additionally say about all odd functions?

1. $a \in D(f) \Rightarrow -a \in D(f)$.
2. The graph of odd function is symmetric with respect to the origin; that is, it has central symmetry.
3. The graph of any odd polynomial function must go through the origin. This is because if both a and $-a$ must belong to the domain, then so must zero.

When a function is odd, sometimes it is not that obvious, so make sure that you do check that $f(-x) = -f(x)$ holds. Otherwise, the function is not odd. Let us consider a function $f(x) = \frac{1}{x-2} + \frac{1}{x+2}$ that was mentioned in the previous section, and let us demonstrate that it is odd:

$$f(-x) = \frac{1}{-x-2} + \frac{1}{-x+2} = -\frac{1}{x+2} - \frac{1}{x-2} = -\left(\frac{1}{x-2} + \frac{1}{x+2}\right) = -f(x)$$

We can also see that its graph is symmetric to the origin (Figure 1.3).

Next, here is a problem for you to try.

Problem 12 It is known that the function $f(x)$ is an odd function defined on the entire real number line and that $x = 4$ is the only positive root of the equation $f(x) = 0$. Find other roots of the equation.

Solution If a function is odd, then $f(-x) = -f(x)$ and then $f(-4) = -f(4)$. However, $f(4) = 0 \Rightarrow f(-4) = 0$. Moreover, each odd function defined on the entire real number line passes the origin. Therefore there are three real zeroes: -4, 0, and 4.

Answer -4, 0, and 4.

Remark Can you give an example of such an odd function? Yes, we can. Assuming that the function is a polynomial of the minimal degree, then, for example, it can be $f(x) = a(x-4)(x+4)x = ax^3 - 16ax$.

In general, a polynomial function that satisfies the problem's condition can be written as $g(x) = a(x-4)(x+4)x^n$ where $a \in R$ and $n = 2k - 1$, $k \in N$, i.e., the natural numbers. Please check this yourself by evaluating $g(-x)$.

1.5 Periodic Functions

Function $y = f(x)$ is called a ***periodic function*** if there exists a number $T > 0$ that is called the *period* of the function, such that for all $x \in D(f)$ the following is true: $x + T \in D(f)$, $x - T \in D(f)$ and $f(x - T) = f(x) = f(x + T)$.

Note that a periodic function is necessarily cyclic on its domain, which can be either infinite or finite. Thus all trigonometric functions are periodic. For example, $\sin(x + 2\pi n) = \sin x$, $n \in Z$. Yes, the simple sine function has infinitely many periods—all multiples of 2π so that the minimal period of the sine function is 2π. This can be proven by contradiction.

Assume that there exists a positive number $l < 2\pi$, such that $\sin(x + l) = \sin x$, $\forall x \in R$. Then at $x = 0$, $\sin(x + l) = 0$ so $\sin(l) = 0$. The number l must be a zero of the function $\sin x$. There is only one zero of the sine function on the interval $(0, 2\pi)$, i.e., π. However, $\sin(x + \pi) \neq \sin x$, e.g., if $x = \frac{\pi}{2}$, then $\sin\left(\frac{\pi}{2}\right) = 1$, but $\sin\left(\frac{\pi}{2} + \pi\right) = -1$, which contradicts the presumption, so π is not a period of the sine function.

You can read more about trigonometric functions in Chapter 3 of the book. Many functions are not periodic, such as polynomial functions or exponential functions. However, there exist functions that are not trigonometric but also periodic. For example, consider the **floor function** $\lfloor x \rfloor$ which is defined as the greatest integer not exceeding x. For example, $\lfloor -3.4 \rfloor = -4$, $\lfloor -4.1 \rfloor = -5$, $\lfloor -4 \rfloor = -4$, and $\lfloor 3.4 \rfloor = 3$. Consider a function $y = f(x) = \{x\}$. By this notation, we will represent a fractional part of number x that is the difference between the number x and its floor $\lfloor x \rfloor$. Thus, $\{x\} = x - \lfloor x \rfloor$.

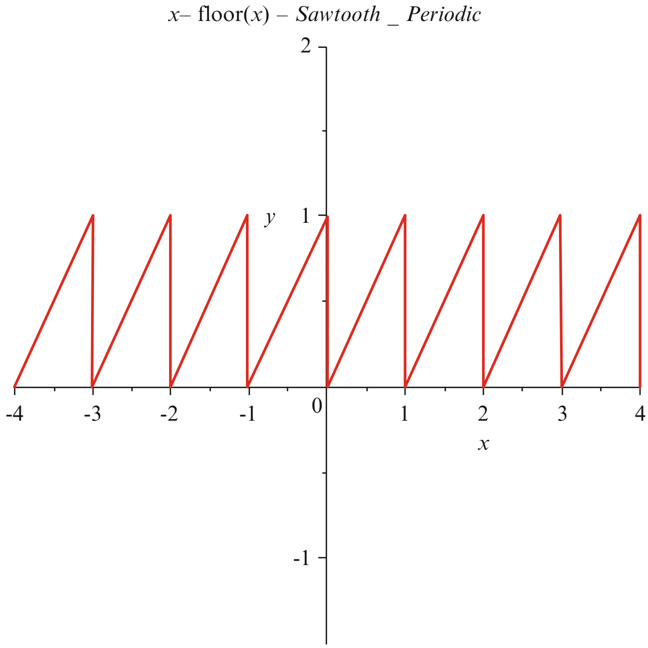


Figure 1.11 Sawtooth function

Next, we can look at some values of this function, $\{3\} = 3 - \lfloor 3 \rfloor = 0$; $\{3.1\} = 3.1 - \lfloor 3.1 \rfloor = 0.1$; $\{-3.4\} = -3.4 - \lfloor -3.4 \rfloor = -3.4 + 4 = 0.6$. It follows from the definition of $y = f(x) = \{x\}$ that its domain is all real numbers and that its range is $y \in [0, 1)$. This function is greater than or equal to zero and also defined by Graham et al. in his book "Concrete Mathematics: A Foundation for Computer Science."

The period of this function is any natural number $T = n,\ n \in N$.

Proof $\{x + n\} = (x + n) - \lfloor x + n \rfloor = x + n - \lfloor x \rfloor - n = x - \lfloor x \rfloor = \{x\}$. The minimal period for this function is $T = 1$. This function $y = f(x) = \{x\}$ is frequently known as the *sawtooth* function because of its look (see Figure 1.11).

In general we can create a **sawtooth function** with period $T = a$, where b is a parameter and $y(x) = b\left(\frac{x}{a} - \left\lfloor \frac{1}{2} + \frac{x}{a} \right\rfloor\right)$. Both functions are shown below (the latter function is shown in Figure 1.12 by the dashed line for $b = 1$, $a = 3$).

Another periodic function with period one $y = g(x) = x - round(x)$ is sketched in Figure 1.13.

Let us list the basic properties of periodic functions.

Property 1 If the function $f(x)$ is periodic with period T, then function $f(kx)$ is also periodic with period $T_1 = \frac{T}{k}$.

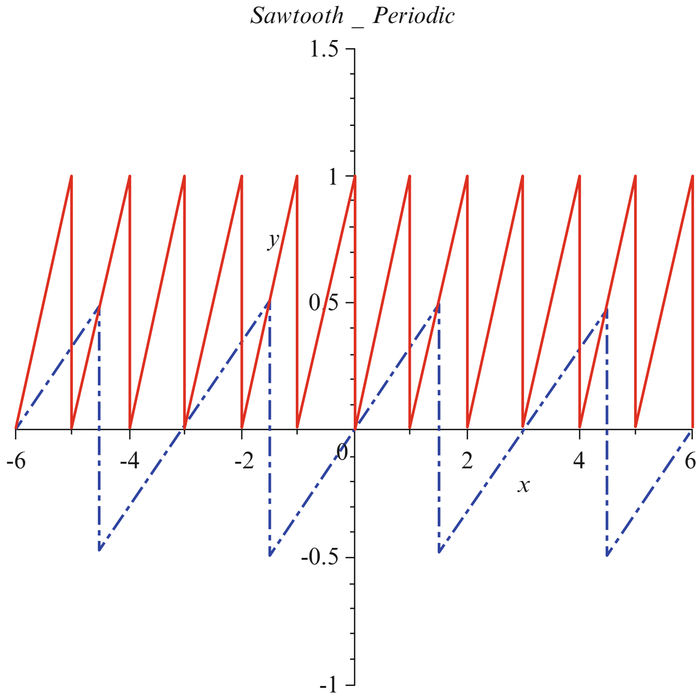


Figure 1.12 Sawtooth functions with periods $T = 1$ and $T = 3$

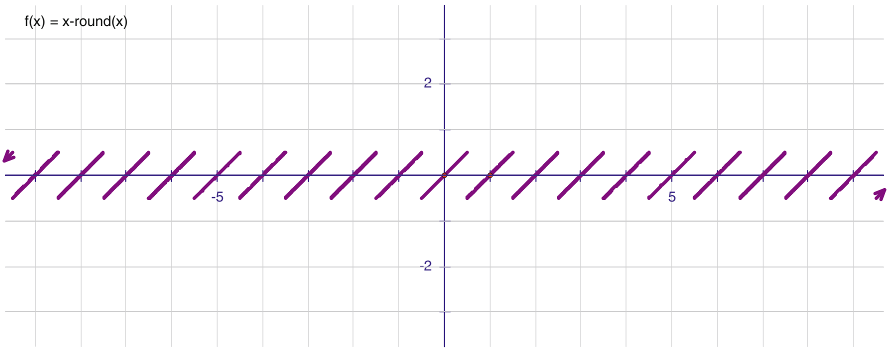


Figure 1.13 Periodic function

Example Function $\cos x$ has period, $T = 2\pi$. Similarly, the function $\cos 3x$ has period $T_1 = \frac{2\pi}{3}$.

Property 2 If a function is the sum or the product of two periodic functions of the same period T, then this function is also periodic with period T. However this number T may not be its minimal period.

Example Both functions $\cos x$ and $\sin x$ are periodic functions with periods $T = 2\pi$. Their product, the function $f(x) = \cos x \sin x$, is also periodic but its minimal period is $T = \pi$.

Property 3 If a function is the sum of periodic functions with different periods, then the function is not necessarily a periodic function.

Problem 13 Find the minimal period of $f(x) = \cos x + \cos 3x$.

Solution Because 2π is the period for both functions of the sum then 2π is the period of the function. Let us show that 2π is also the minimal period. In order to prove this it is enough to prove that 2π is the distance between two neighboring maximums of the functions. If both cosine terms equal one, the function will obtain a maximum at that x value. To determine if this occurs, we attempt to solve the system:

$$\begin{cases} \cos x = 1 \\ \cos 3x = 1 \end{cases} \Leftrightarrow \begin{cases} x = 2\pi n, n \in Z \\ x = \dfrac{2\pi m}{3}, m \in Z \end{cases} \Rightarrow x = 2\pi n, n \in Z.$$

So the distance between two neighboring maxima is 2π.

Answer The minimal period is 2π.

Problem 14 Let $f(x)$ be a periodic function with period $T = \frac{1}{3}$. Evaluate $f(1)$ if $f^2(2) - 5f(0) + \frac{21}{4} = 0$ and $4f^2(-1) - 4f\left(\frac{10}{3}\right) = 35$.

Solution This is an unusual problem and in order to solve it we need to apply the properties of periodic functions. Since $f(x)$ is periodic, $f(x) = f(x + nT)$, where $n \in N$ and $T = \frac{1}{3}$. Replacing n by 1, +1, −1, +2, −2, +3, −3, +4, −4, etc. we obtain

$$f(1) = f\left(1 + \frac{1}{3}\right) = f\left(1 + \frac{2}{3}\right) = f\left(1 + \frac{3}{3}\right) = f(2) = f\left(1 + \frac{7}{3}\right) = f\left(\frac{10}{3}\right)$$
$$= f\left(1 - \frac{3}{3}\right) = f(0) = f\left(1 - \frac{6}{3}\right) = f(-1)$$

From which we can see that $f(1) = f(2) = f(0) = f(-1) = f\left(\frac{10}{3}\right)$. Next, our equations can be written as a system for unknown $f(1)$:

$$\begin{cases} f^2(1) - 5f(1) + \dfrac{21}{4} = 0 \\ 4f^2(1) - 4f(1) = 35 \end{cases} \Leftrightarrow \begin{cases} -4f^2(1) + 20f(1) - 21 = 0 \\ 4f^2(1) - 4f(1) = 35 \end{cases} \Rightarrow$$

$$16f(1) = 56 \Rightarrow f(1) = \frac{7}{2}$$

We can check that this value also satisfies the second equation of the system.

Answer $f(1) = \frac{7}{2}$.

Remark To solve this problem we did not need to know the function itself but we were able to find its value at $x = 1$.

Problem 15 The function $f(x)$ is defined for all real numbers. It is an odd periodic function with period, $T = 4$, and it is defined by the formula $f(x) = 1 - |x - 1|$ on $x \in [0, 2]$. Solve $2f(x) \cdot f(x - 8) + 5f(x + 12) + 2 = 0$.

Solution Because the function has period of four, it is suitable to consider this function on any segment of length four, e.g., $[-2, 2]$. Because our function is odd and we already know its behavior on $[0, 2]$, we can extend it onto $[-2, 0)$ by $f(-x) = -f(x)$ equivalently, so $f(x) = -f(-x) = -(1 - |-x - 1|) = -1 + |x + 1|$, $x \in [-2, 0]$ where there is equality at $x = 0$ for the system of equations:

$$f(x) = \begin{cases} -1 + |x + 1|, \ x \in [-2, 0] \\ 1 - |x - 1|, \ x \in [0, 2] \end{cases}.$$

Together with the periodic condition, $f(x) = f(x + 4n)$ where $n \in Z$ it immediately follows that $f(x) = f(x + 12) = f(x - 8)$. Then the equation that we need to solve will be written as $2f(x)^2 + 5f(x) + 2 = 0$.

This quadratic equation has the solutions:

$$f(x) = -2$$
$$f(x) = -\frac{1}{2}$$

Case 1 $x \in [-2, 0]$

$$\left[\begin{array}{l} -1 + |x + 1| = -2 \\ -1 + |x + 1| = -\dfrac{1}{2} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} |x + 1| = -1 \\ |x + 1| = \dfrac{1}{2} \end{array}\right. \Rightarrow \left[\begin{array}{l} x = -\dfrac{1}{2} \\ x = -\dfrac{3}{2} \end{array}\right.$$

Case 2 $x \in [0, 2]$

$$\left[\begin{array}{l} 1 - |x-1| = -2 \\ 1 + |x-1| = -\dfrac{1}{2} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} |x-1| = 3 \\ |x-1| = \dfrac{3}{2} \end{array}\right. \Rightarrow \left[\begin{array}{l} x \in \{-2, 4\} \\ x \in \left\{-\dfrac{1}{2}, \dfrac{5}{2}\right\} \end{array}\right.$$

We can see that only Case 1 gives us solutions satisfying the restricted segment. We rewrite them in terms of the periodicity of the function
$x = -\frac{1}{2} + 4n$, $x = -\frac{3}{2} + 4m$ where $n, m \in Z$.

Answer $x = -\frac{1}{2} + 4n$, $x = -\frac{3}{2} + 4m$ where $n, m \in Z$.

Problem 16 The function $f(x) = \{x\} = x - \lfloor x \rfloor$ is defined for all real numbers. Solve the equation $\left\{(x+1)^3\right\} = x^3$.

Solution It follows from the definition of the fractional function that

$$0 \le x^3 < 1 \Rightarrow 0 \le x < 1.$$

Expanding the left side of the equation, we can rewrite it as

$$\left\{x^3 + 3x^2 + 3x + 1\right\} = x^3 \text{ or as } \{z + T\} = \{z\}, z = x^3.$$

Because $0 \le z = x^3 < 1$ then the given equation will be true if and only if $p(x) = 3x^2 + 3x = n, n \in Z^+ (n = 0, 1, 2, 3, \ldots)$.

Note that $p(x)$ is increasing since x is increasing on $[0, 1)$. Thus $p(0) = 0$ and $p(1) = 6$.

Then n can take any values between 0 and 6, excluding 6:

$$3x^2 + 3x - n = 0$$
$$x = \frac{-3 \pm \sqrt{9 + 12n}}{6}, n = 0, 1, 2, 3, 4, 5.$$

1.6 Summary of Useful Properties and Their Applications

It is useful to remember the following:

Property 4

$$a + \frac{1}{a} \ge 2 \quad \text{if} \quad a > 0$$
$$a + \frac{1}{a} \le -2 \quad \text{if} \quad a < 0$$

Let us prove Property 4 for $a > 0$. Using the inequality between the arithmetic and geometric means, i.e., $\frac{a+b}{2} \ge \sqrt{ab}$, we have $a + \frac{1}{a} \ge 2\sqrt{a \cdot \frac{1}{a}} = 2$.

Now we will prove the second part of this statement. If $a < 0$, then we can rewrite the left side as $a + \frac{1}{a} = -\left(-a + \frac{1}{-a}\right)$ and apply the now familiar formula to the expression within parentheses. For all $x < 0$, $-x - \frac{1}{x} \geq 2$ so that $x + \frac{1}{x} \leq -2$.

Problem 17 Solve $2^{1-|x|} = 1 + x^2 + \frac{1}{1+x^2}$.

Solution You can notice that the left side of the equation is always positive and decreasing for any x. It is less than or equal to two for any real x. The expression on the right-hand side is positive as well and its lower bound can be estimated as

$$1 + x^2 + \frac{1}{1+x^2} \geq 2\sqrt{(1+x^2)\frac{1}{1+x^2}} = 2$$

where we used the inequality between the geometric and arithmetic means. The solution exists only when the left-hand side equals two. Thus $x = 0$.

Answer $x = 0$.

Property 5 $|\sin x| \leq 1$ and $|\cos x| \leq 1$, $x \in R$.

Let us apply this property for the following problem.

Problem 18 Solve $\cos^{7/5} x + \sin^{5/3} x = \sqrt{2}$.

Solution Since the right side of the given equation is positive, we should consider only such situations when both $\cos x > 0$ and $\sin x > 0$. Then $\cos^{7/5} x < \cos x$ and $\sin^{5/3} x < \sin x$. Adding the left and the right sides of two inequalities, we obtain that $\cos^{7/5} x + \sin^{5/3} x < \cos x + \sin x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) \leq \sqrt{2}$. The left side is always less than the right side, so the equation has no solutions.

Answer No solutions.

Remark In the problem above we used the **Method of the Auxiliary Argument** that allows the user to rewrite $\cos x + \sin x$ as $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$. You can read more about how to employ this method in Chapter 3.

Property 6 A quadratic function with $a \neq 0, f(x) = ax^2 + bx + c$ has a *lower bound* at $f\left(-\frac{b}{2a}\right)$ if $a > 0$ and an *upper bound* at $f\left(-\frac{b}{2a}\right)$ if $a < 0$.

Let us illustrate this statement with an example. After completing the square, the function $f(x) = x^2 - 4x + 9$ can be rewritten as $f(x) = (x-2)^2 + 5$. We can see that $f(2) = 5$ is minimum of $f(x)$. For any other value of x, $f(x) > 5$.

From the properties of quadratic functions we know that the value $x = -\frac{b}{2a}$ is the x coordinate of the vertex of a parabola. If the leading coefficient is positive, $a > 0$, the parabola opens upward, and the quadratic function has a minimum at its vertex, and it will be the lower bound. On the other hand, if $a < 0$, then the parabola opens downward, the quadratic function has a maximum at its vertex, and it will be the upper bound.

Property 7 If $f(x) \geq 0$ and $g(x) \geq 0$, then

$$f(x) + g(x) = 0 \quad \Leftrightarrow \text{(if and only if)} \quad \begin{cases} f(x) = 0 \\ g(x) = 0. \end{cases}$$

The problem below shows how this property can be applied.

Problem 19 Solve the equation: $\sin^2(\pi x) + \sqrt{x^2 + 3x + 2} = 0$.

Solution Notice that the left side of the equation is the sum of two nonnegative functions, but the right side equals 0. This can happen if and only if $\sin^2(\pi x) = 0$ and $\sqrt{x^2 + 3x + 2} = 0$ simultaneously. The first equation yields

$$\begin{aligned} &\pi x = \pi n \\ &x = n, \quad n = 0, \pm 1, \pm 2, \pm 3, .. \end{aligned}$$

The second gives us two roots, $x = -1$ and $x = -2$.

By finding the intersection of the first and the second solutions, we obtain the answer $x = -1$ and $x = -2$.

Answer $x = -1$ and $x = -2$.

Property 8 If $f(x) \geq a$, but $g(x) \leq a$, then

$$f(x) = g(x) \Leftrightarrow \begin{cases} f(x) = a \\ g(x) = a. \end{cases}$$

The following problem will help you to understand this rule.

Problem 20 Solve the equation $1 + x^2 = \cos 3x$.

Solution By Property 6 above, the left side of the equation is greater than or equal to 1, while the right side ($\cos 3x$) by Property 5 is less than or equal to 1. This fits property 8, and therefore yields the system:

$$\begin{cases} 1+x^2=1 \\ \cos 3x = 1 \end{cases} \Leftrightarrow \begin{cases} x=0 \\ 3x=2\pi n \end{cases} \Leftrightarrow \begin{cases} x=0 \\ x=\dfrac{2\pi}{3}\cdot n, \quad n=0,\pm 1,\pm 2,\ldots \end{cases} \Leftrightarrow x=0$$

Answer $x=0$.

Property 9 $|f(x)| \geq a$, but $|g(x)| \geq b$ for $a>0$, $b>0$ and $f(x)$ and $g(x)$ have the same sign, then $f(x)\cdot g(x)=ab \Leftrightarrow \begin{cases} |f(x)|=a \\ |g(x)|=b \end{cases}$.

Let us show how Property 9 can be used.

Problem 21 Does the equation $\left[(x-2)^2+4\right]\cdot\left[x+\frac{1}{x}\right]=8$ have any real solutions?

Solution Instead of trying to simplify the equation, we will look closely at the functions within the brackets. Notice that $\left[(x-2)^2+4\right]\geq 4$, and $\left[x+\frac{1}{x}\right]\geq 2$. On the other hand, $8=4\cdot 2$.

By Property 9 the given equation can have solutions if and only if

$$\begin{cases} (x-2)^2+4=4 \\ x+\dfrac{1}{x}=2 \end{cases} \Leftrightarrow \begin{cases} x=2 \\ x=1 \end{cases}$$

This system has no solutions! Our two functions cannot approach their lower bounds 4 and 2, respectively, at the same value of x.

Answer There are no solutions.

Remark Of course, any equation of the form $\left[(x-2)^2+4\right]\cdot\left[x+\frac{1}{x}\right]=a$, where $a<8$ will have no solutions.

Let us change the previous problem just a little bit and look what will happen if $a>8$.

Problem 22 Find all real solutions of the equation
$\left[(x-2)^2+4\right]\cdot\left[x+\frac{1}{x}\right]=10$.

Solution Though the left side stays the same, the right side, 10, is greater than the product of lower bounds of two functions (8), so Property 9 cannot be applied. However, the given equation can be written as

$$(x^2 - 4x + 8)(x^2 + 1) = 10x, \quad x \neq 0$$
$$x^4 - 4x^3 + 9x^2 - 14x + 8 = 0$$

This polynomial equation has two real solutions $x = 1$ and $x = 2$.

Answer $x = 1$ and $x = 2$.

Note: Read more about polynomial equations in Chapter 2 of the book.

Problem 23 Can you change the previous equation so that it will fit Property 9? Solve that equation.

Solution Yes, we can. For example, the equation $\left[(x-1)^2 + 4\right] \cdot \left[x + \frac{1}{x}\right] = 8$ perfectly fits Property 9 because the functions approach their lower bounds at the same $x = 1$:

$$\begin{cases} (x-1)^2 + 4 = 4 \\ x + \dfrac{1}{x} = 2 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ x = 1 \end{cases}$$

Answer $x = 1$.

Property 10 If $|f(x)| \geq a$, and $|g(x)| \leq b$, $a > 0$, $b > 0$ and $f(x)$ and $g(x)$ have the same sign, then $\frac{f(x)}{g(x)} = \frac{a}{b} \Leftrightarrow \begin{cases} |f(x)| = a \\ |g(x)| = b \end{cases}$.

Problem 24 Solve the equation $\frac{x^2 - 2x + 5}{\sin\left(\frac{\pi x}{2}\right)} = 4$.

Solution Let us complete the square in the numerator of the fraction and obtain

$$\frac{(x-1)^2 + 4}{\sin\left(\frac{\pi x}{2}\right)} = 4.$$

Now by Property 10 we have

$$\begin{cases} (x-1)^2 + 4 \geq 4 \\ \left|\sin\left(\dfrac{\pi x}{2}\right)\right| \leq 1 \end{cases} \Rightarrow \begin{cases} (x-1)^2 + 4 = 4 \\ \sin\left(\dfrac{\pi x}{2}\right) = 1 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ \dfrac{\pi x}{2} = \dfrac{\pi}{2} + 2\pi n \end{cases} \Leftrightarrow x = 1.$$

Answer $x = 1$.

Next, in the problems below, you have to try to recognize a pattern and apply an appropriate rule.

Problem 25 Solve the equation $2(1 + \sin^2(x-1)) = 2^{2x-x^2}$

Solution This is an example of a transcendental equation. Using the boundedness of the functions we can solve it in two steps.

Step 1: Consider the given equation as $g(x) = f(x)$. We are looking for common points of the left and right sides.

Next, $g(x) = 2(1 + \sin^2(x-1))$ is continuous for all real x and has the lower bound $g(x) = 2$ and the upper bound $g(x) = 4$ because $0 \le \sin^2(x-1) \le 1$, so $2 \le g(x) \le 4$.

On the other hand, $f(x) = 2^{2x-x^2}$ is bounded above (has the upper bound) because $2x - x^2 = -(x-1)^2 + 1$ has its maximum at $x = 1$, and then $0 < f(x) \le 2$.

Step 2: Putting the equation and the two inequalities together, we can solve the following system:

$$\begin{cases} 2 \le g(x) \le 4 \\ 0 < f(x) \le 2 \\ g(x) = f(x) \end{cases}$$

Therefore the system has a solution if and only if $g(x) = f(x) = 2$. This occurs at $x = 1$.

Answer $x = 1$.

I hope you developed a "taste" for such problems and can recognize them now. So I offer you the following problem.

Problem 26 Solve the equation $\log_{0.5}(\tan \pi x + \cot \pi x) = 8(2x^2 + 3x + 1)$.

Solution A transcendental equation again? Well, it can be written as

$$g(x) = f(x)$$

The argument of the logarithmic function is the bounded function can be analyzed using the inequality between the geometric and arithmetic means:

$$\tan \pi x + \cot \pi x \geq 2\sqrt{\tan \pi x \cdot \cot \pi x} = 2$$

Because $\log_{0.5} u$ is decreasing over the entire domain for all $u > 0$, it will decrease for $u = \tan \pi x + \cot \pi x$ and $g(x) = \log_{0.5}(\tan \pi x + \cot \pi x) \leq \log_{0.5} 2 = -1$. We obtained that $g(x) \leq -1$. On the other hand, by completing the square on the right-hand side of the given equation, we have

$$8(2x^2 + 3x + 1) = 8\left[2\left(x^2 + 2 \cdot \frac{3}{4}x + \frac{9}{16}\right) - \frac{1}{8}\right]$$
$$f(x) = 16\left(x + \tfrac{3}{4}\right)^2 - 1$$

We can see that $f(x)$ has the minimum value -1 or the lower bound at $x = -3/4$ (Property 6). Thus $f(x) \geq -1$.

These can also be combined into the system:

$$\begin{cases} g(x) \leq -1 \\ f(x) \geq -1 \\ g(x) = f(x) \end{cases}$$

This system has a solution iff (if and only if) $g(x) = f(x) = -1$ and $x = -3/4 = -0.75$.

Answer $x = -0.75$.

Problem 27 Does the equation $2^{\sin^2 x} + 2^{\cos^2 x} = 1.5(\tan x + \cot x)$ have any solutions?

Solution Let us rewrite the equation in the form $f(x) = g(x)$. Applying the formula $a + b \geq 2\sqrt{ab}$ to the left and right sides of the equation and replacing $\cos^2 x = 1 - \sin^2 x$ in the exponent, we obtain the lower bounds for both sides of the equation

$$f(x) = 2^{\sin^2 x} + 2^{\cos^2 x} = 2^{\sin^2 x} + \frac{2}{2^{\sin^2 x}} \geq 2\sqrt{2}$$
$$g(x) = 1.5(\tan x + \cot x) \geq 3$$

On the other hand, $f(x)$ has an upper bound as well. Since $0 \leq \sin^2 x \leq 1$, then $f(x) \leq 3$. We can unite these into the system:

$$\begin{cases} 2\sqrt{2} \leq f(x) \leq 3 \\ g(x) \geq 3 \\ f(x) = g(x) \end{cases} \Leftrightarrow g(x) = f(x) = 3$$

Are we getting closer? Can $g(x)$ and $f(x)$ approach the value of 3 simultaneously?

We should mention that $f(x)$ approaches its upper bound, 3, when $\sin^2 x = 1$ or $\sin^2 x = 0$. The latter conditions take place if $x = \frac{\pi}{2} + \pi n$ or $x = \pi k$, $n, k \in Z$, respectively.

However, our function $g(x)$ is undefined for all these values of x.

Answer This equation has no solutions.

Problem 28 Solve the equation
$\sqrt{3x^2 + 6x + 7} + \sqrt{5(x^2 + 2x + 1) + 9} = 4 - 2x - x^2$.

Solution Noticing a trinomial square within the second radicand, let us complete the squares under the first root and on the right-hand side. This yields

$$\begin{array}{ccccc} \sqrt{3(x+1)^2+4} & + & \sqrt{5(x+1)^2+9} & = & 5-(x+1)^2 \\ \geq 2 & + & \geq 3 & = & \leq 5 \end{array}$$

Because $(x+1)^2 \geq 0$ then $\sqrt{3(x+1)^2+4} \geq \sqrt{4} = 2$, $\sqrt{5(x+1)^2+9} \geq \sqrt{9} = 3$, and $5 - (x+1)^2 \leq 5$. Thus, the left side is greater than or equal to 5 but the right side is less than or equal to 5. Now, using property 5 we conclude that such an equation has a solution only when both sides equal 5. This happens when $x + 1 = 0$ or $x = -1$.

Answer $x = -1$.

In order to solve the following and many other problems we need to recall properties of exponential functions.

Property 11 Property of exponential functions:
Consider an exponential function $f(x) = a^x$. Depending on the values of the base and the power, the following is true:

$$\begin{array}{l} a^b > 1 \quad \text{if} \quad a > 1 \text{ and } b > 0 \\ 0 < a^b < 1 \quad \text{if} \quad a > 1 \text{ and } b < 0 \end{array}$$

Problem 29 Solve the inequality $(x^2 + 2x + 2)^x \geq 1$.

Solution Completing the square on the left-hand side, we obtain

$$\left((x+1)^2+1\right)^x \geq 1$$

a. If $x=-1$, then $1^{-1} \geq 1$ is true.
b. If $x \neq -1$, then $(x+1)^2+1>1$ for any $x>0$.

The function on the left can be written as a^x, where $a=(x+1)^2+1>1$. Therefore, a^x is increasing over the entire domain. Moreover, $a^x>1$ if $x>0$. Combining both cases we get the answer below.

Answer $x=-1$ or $x>0$.

Problem 30 For what real values of p does the equation $4^x+2^{x+2}+7=p-4^{-x}-2\cdot 2^{1-x}$ have a solution?

Solution Let us rewrite the equation in a different form by moving all terms except p to the left-hand side and using the properties of exponents:

$$4^x+4^{-x}+4\cdot 2^x+4\cdot 2^{-x}+7=p$$

And then combining the first two and the next two terms

$$(4^x+4^{-x})+4\cdot(2^x+2^{-x})+7=p.$$

Because $a^x>0$ and applying Property 4 to the expressions within parentheses, we notice that the left side of this equation is greater than or equal to $2+4\cdot 2+7=17$.

Now we can conclude that because $f(x)=4^x+4^{-x}+4\cdot 2^x+4\cdot 2^{-x}+7\geq 17$ then the equation $f(x)=p$ has solutions only if $p\geq 17$.

Answer The equation will have solutions for any values of $p\geq 17$ or $p\in[17,\infty)$.

Remark In Problem 30 we did not find the solution of the equation but just described the values of the parameter p for which such a solution exists. Please look in Section 2.4 for the complete solution to this problem (Problem 75).

Problem 31 Solve the inequality $(2^x+3\cdot 2^{-x})^{2\log_2 x-\log_2(x+6)}>1$.

Solution At first glance, this problem seems very difficult.

Applying the inequality between arithmetic and geometric mean (see Property 1), for the base of the exponent on the left side of the inequality we obtain that $2^x+3\cdot 2^{-x}\geq 2\sqrt{\frac{2^x\cdot 3}{2^x}}=2\sqrt{3}>1$. On the right-hand side, we have 1.

Because the base of the exponential function on the left $(2^x + 3 \cdot 2^{-x})$ is greater than 1, it is enough to require that the power $(2\log_2 x - \log_2(x+6))$ be greater than 0, and the given inequality will be always true. Because logarithmic functions have a restricted domains ($x > 0$ and $x + 6 > 0$), we will also add a second inequality to our system:

$$\begin{cases} 2\log_2 x - \log_2(x+6) > 0 \\ \text{Domain}: \quad x > 0 \end{cases} \Leftrightarrow \begin{cases} \log_2\left(\dfrac{x^2}{x+6}\right) > 0 \\ x > 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{x^2}{x+6} > 1 \\ x > 0 \end{cases}$$

We can solve the first inequality by moving 1 to the left side and then putting two fractions over the common denominator:

$$\begin{cases} \dfrac{x^2 - x - 6}{x+6} > 0 \\ x > 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{(x+2)(x-3)}{(x+6)} > 0 \\ x > 0 \end{cases} \Leftrightarrow \begin{cases} x > 3 \\ x > 0 \end{cases} \Leftrightarrow x > 3$$

Remark In order to solve the first inequality, we put zeros of the numerator and the denominator on the number line in increasing order, and then find the signs of the function $f(x) = \frac{(x+2)(x-3)}{(x+6)}$ on each interval, $x < -6$, $-6 < x < -2$, $-2 < x < 3$, and $x > 3$.

We chose those intervals where $f(x)$ is positive ($-6 < x < -2$ and $x > 3$). This method is often called the **Interval Method**: The intersection of these intervals and $x > 0$ is $x > 3$.

Answer $x > 3$.

Problem 32 Solve the equation $\sin^6 x + \cos^{20} x = 1 + \cos^2 x$.

Solution Because a function a^t is decreasing over all real t if $0 < a < 1$, then $\sin^6 \leq \sin^2 x$ and $\cos^{20} x \leq \cos^2 x$. Next, $\sin^6 x + \cos^{20} x \leq \sin^2 x + \cos^2 x = 1$. We have determined that the left side of the equation is less than or equal to 1. On the other hand, the right side of the equation is greater than or equal to 1: $1 + \cos^2 x \geq 1$. Next, by Property 8, in order to have solutions, both sides of the equation must be equal to 1. The right-hand side equals 1 when $\cos x = 0, x = \frac{\pi}{2} + \pi n, n \in Z$. The same values of $\boldsymbol{x}$ make the left side equal to 1. Here set Z is the set of all integers, i.e., $Z = 0, \pm 1, \pm 2, \pm 3, \ldots$.

Answer $x = \frac{\pi}{2} + \pi n, n \in Z$.

Problem 33 Solve the equation $\log_2(3 - \sin x) = \sin x$.

Solution Because $|\sin x| \le 1$ for any real $\boldsymbol{x}$, then $3 - \sin x \ge 2\ \forall x \in R$, and $\log_2(3 - \sin x) \ge \log_2 2 = 1$.

From this we can see that the left side is always greater or equal to 1. Simultaneously, the right side of the given equation is less than or equal to one. ($\sin x \le 1$) The equation will have solutions if and only if both sides are equal to 1. This yields $x = \frac{\pi}{2} + 2\pi n, \quad n \in Z$.

Answer $x = \frac{\pi}{2} + 2\pi n, \quad n \in Z$.

Problem 34 Solve the equation $|\sin 3x|^{\cot 2x} = 1$

Solution To solve this equation we need to consider two cases:

a. If $\sin 3x \ne 0$, then $\cot 2x = 0$

$$2x = \frac{\pi}{2} + \pi n, \text{ and } x = \frac{\pi}{4} + \frac{\pi}{2} \cdot n, \quad n = 0, \pm 1, \pm 2, \ldots.$$

b. If $\sin 3x = \pm 1$, then

$$3x = \frac{\pi}{2} + \pi k, \text{ and } x = \frac{\pi}{6} + \frac{\pi}{3} \cdot k, \quad k = 0, \pm 1, \pm 2, \ldots.$$

Answer $x = \frac{\pi}{4} + \frac{\pi}{2} \cdot n, \quad n = 0, \pm 1, \pm 2, \ldots$
or $x = \frac{\pi}{6} + \frac{\pi}{3} \cdot k, \quad k = 0, \pm 1, \pm 2, \ldots.$

Remark See more about solving trigonometric equations in Chapter 3 of the book.

Problem 35 $f(x)$ satisfies the equation $(x - 1) f\left(\frac{x+1}{x-1}\right) - f(x) = x, \quad \forall x \ne 1.$ Find all such functions.

Solution If the given equation works for $\forall x \ne 1$, then it will work for $x \to \frac{x+1}{x-1}$. We have the following:

$$\left(\frac{x+1}{x-1} - 1\right)\left(f\left(\frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1}\right) - f\left(\frac{x+1}{x-1}\right)\right) = \frac{x+1}{x-1}$$

Simplifying expressions inside the parentheses we obtain that

$$\frac{x+1-x+1}{x-1}\cdot f\left(\frac{x+1+x-1}{x+1-x+1}\right)-f\left(\frac{x+1}{x-1}\right)=\frac{x+1}{x-1}$$
$$\frac{2}{x-1}f(x)-f\left(\frac{x+1}{x-1}\right)=\frac{x+1}{x-1}$$

Solving this equation together with the original equation, we can find the function from the system:

$$\begin{cases}\frac{2}{x-1}f(x)-f\left(\frac{x+1}{x-1}\right)=\frac{x+1}{x-1}\\(x-1)f\left(\frac{x+1}{x-1}\right)-f(x)=x\end{cases}$$

From the first equation, we obtain that $f\left(\frac{x+1}{x-1}\right)=\frac{2f(x)}{x-1}-\frac{x+1}{x-1}$.

Substituting this value into the second equation, we have $(x-1)\left[\frac{2f(x)}{x-1}-\frac{x+1}{x-1}\right]-f(x)=x$, from which we finally have $f(x)=1+2x$.

Answer $f(x)=1+2x$.

1.7 Homework on Chapter 1

1. Prove that the function $f(x)=\sqrt{x}$ is increasing over its entire domain, $D(f)=[0,+\infty)$.
   Proof: Let $x_2>x_1$, and $(x_1,x_2\in D(f))$, and consider the difference $f(x_2)-f(x_1)$ and transform it by multiplying the numerator and denominator by the positive factor $\left(\sqrt{x_2}+\sqrt{x_1}\right)$:

$$f(x_2)-f(x_1)=\sqrt{x_2}-\sqrt{x_1}=\frac{\left(\sqrt{x_2}+\sqrt{x_1}\right)\left(\sqrt{x_2}-\sqrt{x_1}\right)}{\left(\sqrt{x_2}+\sqrt{x_1}\right)}$$
$$=\frac{x_2-x_1}{\sqrt{x_2}+\sqrt{x_1}}>0\Rightarrow f(x_2)>f(x_1)$$

   The proof is completed.
2. Is the function $f(x)=\sqrt{x+1}-\sqrt{x-1}$ increasing or decreasing?
   Answer: The function is decreasing on its domain: $x\geq 1$ because it can be written as $f(x)=\frac{2}{\sqrt{x+1}+\sqrt{x-1}}$ that is always less than another monotonically decreasing function $g(x)$:

$$f(x) = \frac{2}{\sqrt{x+1}+\sqrt{x-1}} < \frac{2}{2\sqrt{x+1}} = g(x) = \frac{1}{\sqrt{x+1}}$$
$$f(x) < g(x)$$

3. Prove for all nonnegative values a and b that the following inequality is true: $(a+1)(b+1)(ab+1) \geq 8ab$.
Proof: Using the inequality between arithmetic and geometric means we have the following correct inequalities:

$$(a+1) \geq 2\sqrt{a}$$
$$(b+1) \geq 2\sqrt{b}$$
$$(ab+1) \geq 2\sqrt{ab}$$

Multiplying positive left and right sides, we get the required inequality:

$$(a+1)\cdot(b+1)\cdot(ab+1) \geq 2\sqrt{a}\cdot 2\sqrt{b}\cdot 2\sqrt{ab} = 8ab.$$

4. How many times does the line $y = 12$ intersect the graph of function $f(x) = \sqrt{x+3}+\sqrt{x+10}+\sqrt{x+19}$? Find coordinates for all the points of intersection.
Solution: Function $f(x)$ is the sum of three monotonically increasing functions, so it takes each y value only one time. Therefore, it can have only one intersection with the line $y = 12$ (see Figure 1.14). We can find that $x = 6$ is the solution of $\sqrt{x+3}+\sqrt{x+10}+\sqrt{x+19} = 12$, so $x = 6$ is the only solution to this equation.
Answer: (6, 12).
5. It is known that the function $f(x)$ is an odd function defined on the entire real number line. It is also known that $0 \in D(f)$ and $x = 5, x = -2$ are zeros of the function. Find other possible zeros of the function.
Hint: See Problem 12.
Answer: $x \in \{-5, -2, 0, 2, 5\}$.

6. If $f(x) = x^8 + px^4 + 1$ and $f(2) = 305$, find $f(-2)$ and the value of p.
Solution: The function is even, so $f(-2) = f(2) = 305$. Next, with $a = 2$ we get

$$f(x) = x^8 + px^4 + 1$$
$$p = \frac{f(a) - 1 - a^8}{a^4} = \frac{305 - 1 - 256}{16} = 3$$

Answer: $p = 3,\ f(-2) = 305$.

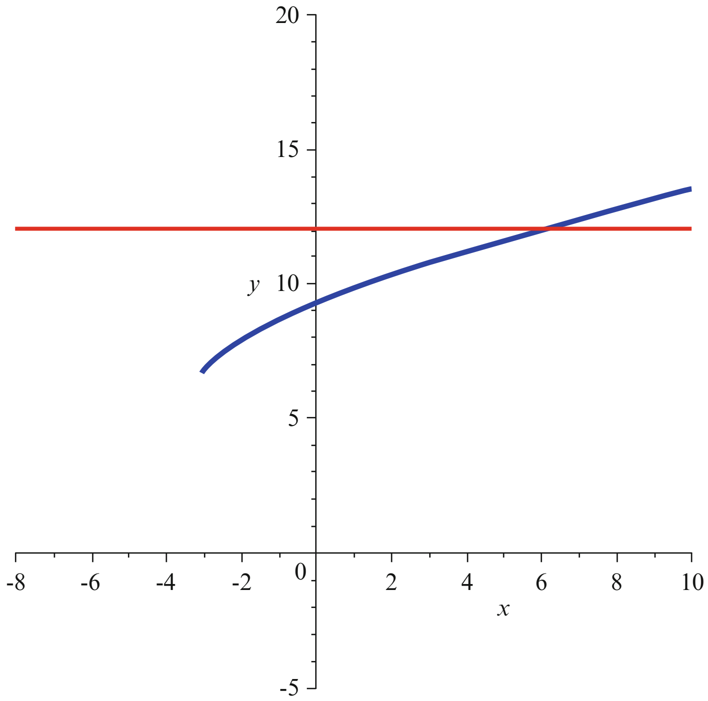


Figure 1.14 HW Problem 4 (Chapter 1)

7. Solve the equation $\sqrt[8]{x+1}+\sqrt[8]{x-1}=\sqrt[8]{2}$
   Solution: Because the function on the left is monotonically increasing, this equation can have only one solution.
   Answer: $x = 1$.

8. Solve the equation $\sqrt{1+x+x^2}+\sqrt{1+x^2-x}=2$.
   Solution: Completing the square inside each root we obtain
$$\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}+\sqrt{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}=2$$
   Answer: $x = 0$.

9. Prove that the sum of two increasing functions is increasing and the sum of two decreasing functions is a decreasing function.
   Proof: Let $\varphi(x) = f(x) + g(x)$ where $D(\varphi) = D(f) \cap D(g)$ and let both x_1 and x_2 belong to the domain of the function $\varphi(x)$ and also satisfy $x_2 > x_1$.
   Consider the difference
$$\begin{aligned}\varphi(x_2) - \varphi(x_1) &= f(x_2) + g(x_2) - (f(x_1) + g(x_1)) \\ &= (f(x_2) - f(x_1)) + (g(x_2) - g(x_1))\end{aligned}$$

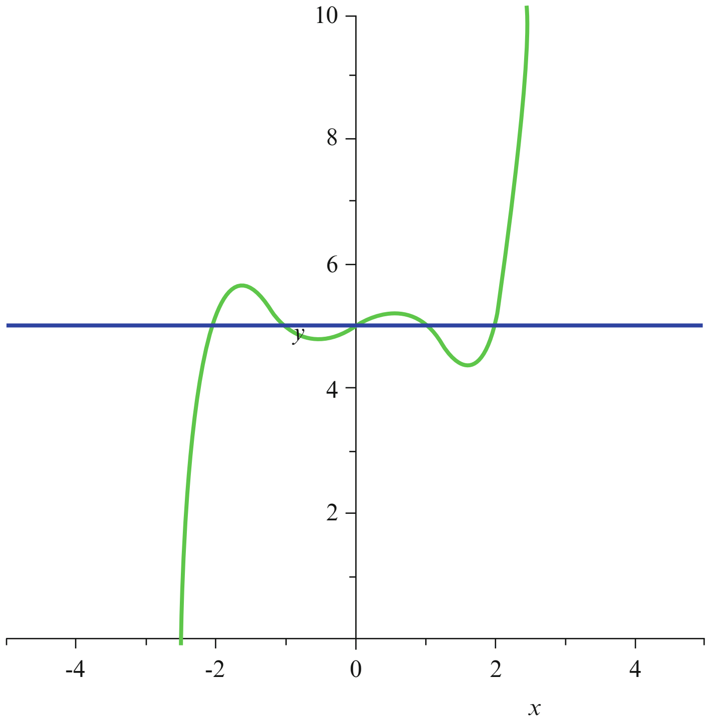


Figure 1.15 Sketch for HW Problem 10 (Chapter 1)

If both $f(x)$ and $g(x)$ are increasing functions, then $f(x_2) > f(x_1)$ and $g(x_2) > g(x_1)$. Hence $f(x_2) - f(x_1) > 0$ and $g(x_2) - g(x_1) > 0$. Therefore $\varphi(x_2) - \varphi(x_1) > 0$ and $\varphi(x)$ is an increasing function.

If both $f(x)$ and $g(x)$ are decreasing functions, then $f(x_2) < f(x_1)$ and $g(x_2) < g(x_1)$. Hence $f(x_2) - f(x_1) < 0$ and $g(x_2) - g(x_1) < 0$. Therefore $\varphi(x_2) - \varphi(x_1) < 0$ and $\varphi(x)$ is a decreasing function.

The proof is completed.

10. Give an example of a function that is continuous on $[-5, 5]$ and takes the value of 5 exactly 5 times (see Figure 1.15).
Answer: $f(x) = \frac{(x+2)(x+1)x(x-1)(x-2)}{\sqrt{36-x^2}} + 5$.

11. Solve the equation $\sqrt{x} + x^3 - \frac{2}{x} = 0$.
Solution: Consider the function of the left-hand side $f(x) = \sqrt{x} + x^3 - \frac{2}{x}$. It is defined for all positive values of x. Also it consists of three monotonically increasing functions, $\sqrt{x}$, x^3, and $-\frac{2}{x}$. (The last one is increasing because it is a reciprocal of the monotonically decreasing function $y = -\frac{x}{2}$.) Because any monotonically increasing function has at most one zero and $x = 1$ satisfies the equation, then it is the only solution to the equation.
Answer: $x = 1$.

12. Prove that these functions are decreasing functions and find their domains:

 a. $f(x) = \sqrt{4-x}$
 b. $g(x) = \frac{1}{\sqrt{4+x}}$

 Proof:

 a. Because the first function is the composition of an increasing, $\varphi(x) = \sqrt{x}$, and decreasing function, $\psi(x) = 4 - x$, then by Theorem 6 it is a decreasing function.
 b. By Theorem 7, the given function is a decreasing function as a reciprocal of an increasing function, $\varphi = \sqrt{4+x}$.

13. Prove that the functions are increasing functions:

 a. $f(x) = \sqrt{3x} - \frac{5}{\sqrt{3x}}$
 b. $g(x) = \frac{5x+3}{3-x}$, $x \in (3, \infty)$

 Hint:

 a. Apply Theorems 2, 7, and 3. The first function is an increasing function as the sum of two increasing functions.
 b. Apply Theorem 4: $g(x) = (5x+3) \cdot \frac{1}{3-x}$ is an increasing function on the interval $x > 3$ as a product of two positive increasing functions.

14. Prove Theorem 6 that if $f(x)$ is a decreasing function and $g(x)$ is an increasing function, then their composition function $h(x) = (f \circ g)(x)$ is a decreasing function.
 Hint: The proof is similar to that of Theorem 5.
 Proof: If $g(x)$ is increasing, then $\forall x_2 > x_1$, and $g(x_2) > g(x_1)$. Next, because $f(x)$ is decreasing, then $f(g(x_1)) > f(g(x_2))$. Hence $h(x_1) > h(x_2)$. Therefore $h(x) = (f \circ g)(x) = f(g(x))$ is a decreasing function.

15. Solve the equation $x^2 + |x| + \sqrt{x} + 2x = 111$.
 Solution: First, because of the square root, the solution must belong to $x \geq 0$ and the function $f(x) = x^2 + |x| + \sqrt{x} + 2x - 111$ is monotonically increasing. Hence, the given equation will have only one solution. We can see that $x = 9$ satisfies the equation.
 Answer: $x = 9$.

16. Find the type of monotonic behavior of the functions.

 a. $y = \frac{1}{x+3} - \sqrt{x+1}$
 b. $y = \frac{1}{\sqrt{x+3}} + \frac{1}{\sqrt{x-3}}$

 Hint: Apply Theorem 3 and then Theorem 7.
 Answer: Both functions are decreasing functions.

17. Solve the system of equations: $\begin{cases} \sqrt{x-y} + (x-y)^3 = 2 \\ x^2 - 6y + 1 = 0 \end{cases}$

Hint: Use properties of monotonic functions.

Solution: On the one hand, the quantity under the square root must be nonnegative, so that $x - y \geq 0 \Rightarrow x \geq y$. On the other hand, the function on the left-hand side of the first equation of the system is increasing, so that $x - y = 1$ is the only solution to the first equation. Substituting $x = y + 1$ into the second equation of the system and after simplification we obtain a quadratic equation in y: $y^2 - 4y + 2 = 0$. Selecting the only root that is less than 1, we obtain our answer.

Answer: $(x, y) = \left(3 - \sqrt{2}; 2 - \sqrt{2}\right)$.

18. Solve the equation $x^2 + 2x + \sqrt{x+1} - \frac{12}{x+1} = 14$.

Solution: We can rewrite this equation as

$$x^2 + 2x + 1 + \sqrt{x+1} - \frac{12}{x+1} = 14 + 1$$
$$f(x) = (x+1)^2 + \sqrt{x+1} - \frac{12}{x+1} = 15$$
$$f(x) = 15$$

Because the function $f(x)$ is increasing, then it takes each value only once. We notice that $f(3) = 15$, then $x = 3$ is the only solution to this equation.

Answer: $x = 3$.

19. Prove that if $\psi(x)$ with the domain symmetric with respect to zero, then

a. $f(x) = \frac{\psi(x)+\psi(-x)}{2}$ is an even function.
b. $f(x) = \frac{\psi(x)-\psi(-x)}{2}$ is an odd function.

20. Using properties of monotonic functions solve $\frac{x^4+5x-12}{x} = 7$.

Solution: Because $x = 0$ is not in the domain of the function on the left, then we will divide the numerator by x obtaining $\frac{x^4+5x-12}{x} = x^3 + 5 - \frac{12}{x} = 7$ which is equivalent to $x^3 - \frac{12}{x} = 2$. The function on the left is the sum of two increasing functions, so it will be monotonically increasing on the domain and it will take each value once. Therefore $x = 2$ is the only solution to this equation.

Answer: $x = 2$.

21. Solve the equation $\sqrt{x^7+1} + \sqrt{1-x^5} = 3$.

Solution: Let us find the interval for all possible values of x as restricted by the domains of the square root functions. We obtain $|x| \leq 1$. Then the following is true:

$$0 \leq \sqrt{x^7+1} \leq \sqrt{2}$$
$$0 \leq \sqrt{1-x^5} \leq \sqrt{2}$$

and the left side of the equation is bounded, $f(x) = \sqrt{x^7+1} + \sqrt{1-x^5} \leq 2\sqrt{2} = \sqrt{8}$. Because $\sqrt{8} < \sqrt{9} = 3$ (the right side of the equation), then there are no solutions.

Answer: No solutions.

22. Solve the equation $\sqrt{4-x} + \sqrt{x-2} = x^2 - 6x + 11$.

Solution: After completing the square on the right-hand side we obtain $(x-3)^2 + 2 \geq 2$, so the function on the right has the lower boundary of 2. Let us show that the function on the left is less than or equal to 2:

$$\begin{aligned}
&\sqrt{4-x} + \sqrt{x-2} = f(x) > 0\\
&4 - x + x - 2 + 2\sqrt{(4-x)(x-2)} = f^2(x)\\
&2 + 2\sqrt{(4-x)(x-2)} = f^2(x)\\
&f^2(x) = 2 + 2\sqrt{-(x-3)^2 + 1} \leq 4 \Rightarrow f(x) \leq 2\\
&f(3) = 2
\end{aligned}$$

Answer: $x = 3$.

23. Find the maximum and minimum of the function $y = \frac{2x^2}{x^4+1}$.

Hint: Divide the numerator and denominator by x^2: $y = \frac{2}{x^2+\frac{1}{x^2}}$. Because $x^2 + \frac{1}{x^2} \geq 2 \Rightarrow y \leq 1$.

Answer: $\max\left(\frac{2x^2}{x^4+1}\right) = 1,\ \min\left(\frac{2x^2}{x^4+1}\right) = 0$.

24. Find the maximum and minimum of $y = \frac{x^2+1}{x^2+x+1}$.

Solution: The function can be rewritten as

$$y = \frac{x^2+x+1}{x^2+x+1} - \frac{x}{x^2+x+1} = 1 - \frac{1}{x + \frac{1}{x} + 1}.$$

Therefore, we need to consider two cases:
Case 1 If $x > 0 \Rightarrow x + \frac{1}{x} \geq 2 \Rightarrow x + \frac{1}{x} + 1 \geq 3 \Rightarrow \frac{1}{x+\frac{1}{x}+1} \leq \frac{1}{3}$, then $y \geq 1 - \frac{1}{3} = \frac{2}{3}$, $y_{\min} = \frac{2}{3}$.
Case 2 If $x < 0 \Rightarrow x + \frac{1}{x} \leq -2 \Rightarrow x + \frac{1}{x} + 1 \leq -1 \Rightarrow \frac{1}{x+\frac{1}{x}+1} \geq -1$, then

$y \leq 1 - (-1) = 2 \Rightarrow y_{\max} = 2$.

Answer: $y_{\min} = \frac{2}{3}$; $y_{\max} = 2$.

25. Solve the equation $\sqrt{16 - 8x + x^2} + \sqrt{4x^2 - 13x - 17} = x - 4$.

Solution: Because the left side is nonnegative, then $x - 4 \geq 0 \Rightarrow x \geq 4$. Completing the square under the first square root and using the inequality above, we obtain $\sqrt{16 - 8x + x^2} = \sqrt{(4-x)^2} = |4 - x| = x - 4$. Now the equation will be rewritten as

$$\begin{cases} \sqrt{4x^2 - 13x - 17} = 0 \\ x \geq 4 \end{cases}$$

Solving this we will obtain the answer $x = 4.25$.

Answer: $x = 4.25$.

26. Solve the equation $2\sin x = y + \frac{1}{y}$.

Solution: Using the boundedness of the left and right sides, there are two cases:

Case 1 $y > 0$: $\begin{cases} \sin x = 1 \\ y = 1 \end{cases} \Leftrightarrow x = \frac{\pi}{2} + 2\pi n;\, y = 1.$

Case 2 $y < 0$: $\begin{cases} \sin x = -1 \\ y = -1 \end{cases} \Leftrightarrow x = -\frac{\pi}{2} + 2\pi m;\, y = -1.$

Answer: $\left(-\frac{\pi}{2} + 2\pi n; -1\right); \left(\frac{\pi}{2} + 2\pi m; 1\right)$.

27. Solve the equation $x^2 + 4x \cdot \cos(xy) + 4 = 0$.

Solution: If we consider this equation as quadratic in x, then for real solutions its discriminant must be nonnegative: $\frac{D}{4} = 4\cos^2 xy - 4 \geq 0$. Because $|\cos xy| \leq 1$, then we will have real solutions if $|\cos xy| = 1$.

1. $\begin{cases} \cos xy = 1 \\ xy = 2\pi n \\ x^2 + 4x + 4 = 0 \end{cases} \Rightarrow x = -2, y = \pi n$

2. $\begin{cases} \cos xy = -1 \\ xy = \pi k \end{cases} \Rightarrow x = 2, y = \frac{\pi}{2} \cdot k$

Answer: $(-2, \pi n); \left(2, \frac{\pi k}{2}\right), k, n \in Z$.

28. Solve the inequality $\left|\tan^3 x\right| + \left|\cot^3 x\right| \leq 2 - \left(x - \frac{\pi}{4}\right)^2$.

Solution: Because $\tan x = \frac{1}{\cot x}$, then $\left|\tan^3 x\right| + \left|\cot^3 x\right| \geq 2$. On the other hand, the right side is less than or equal to 2. Therefore, the solution occurs if both sides are equal to 2.

Answer: $x = \frac{\pi}{4}$.

29. Solve the system $\begin{cases} x^5 + y^5 = 1 \\ x^6 + y^6 = 1 \end{cases}$.

Solution: Consider the second equation of the system; it follows that it is possible if $|x| \leq 1, |y| \leq 1$. From the first equation we obtain $\left|y^5\right| = \left|1 - x^5\right| \leq 1$ and thus we have two solutions.

Answer: (0, 1); (1, 0).

30. Solve the equation $\sin^2(\pi x) + \sqrt{x^2 + 3x + 2} = 0$.

Solution: In order for this equation to have a solution, each term must be zero. Thus

$$\begin{cases} \sin^2 \pi x = 0 \\ x^2 + 3x + 2 = 0 \end{cases} \Rightarrow \begin{cases} \pi x = \pi n \\ (x+1)(x+2) = 0 \end{cases} \Leftrightarrow \begin{cases} x = n, \quad n \in Z \\ x = -1; x = -2 \end{cases}$$

Answer: $x \in \{-1, 2\}$.

31. Solve the inequality $\sqrt{x^2 - 36} \cdot (x+4) < 0$.

Solution: The product of two quantities is positive when both quantities are positive or both are negative. The first factor is nonnegative, so we will have the following system:

$$\begin{cases} x^2 - 36 > 0 \\ x + 4 > 0 \end{cases} \Rightarrow \begin{cases} (x-6)(x+6) > 0 \\ x > -4 \end{cases} \Leftrightarrow \left[\begin{array}{l} \begin{cases} x < -6 \\ x > -4 \end{cases} \\ \begin{cases} x > 6 \\ x > -4 \end{cases} \end{array} \right.$$

Answer: $x > 6$.

32. Solve the inequality $\sqrt[4]{76 + 5^{5^{1-\cos x}}} \leq \sqrt{5 \cdot 2^{-2x^2} - 1}$.

Solution: Using boundedness of the function $y = \cos x$ and monotonic behavior of $y = a^x$, we know that $1 - \cos x \geq 0 \Rightarrow 5^{1-\cos x} \geq 5^0 = 1 \Rightarrow 5^{5^{1-\cos x}} \geq 5$ (bounded below), then $\sqrt[4]{76 + 5^{5^{1-\cos x}}} \geq \sqrt[4]{76 + 5} = 3$. On the other hand, the expression under the square root on the right-hand side is bounded above and

$$\begin{aligned} &-2x^2 \leq 0 \Rightarrow 2^{-2x^2} \leq 1 \Rightarrow 5 \cdot 2^{-2x^2} \leq 5 \\ &\Rightarrow \sqrt{5 \cdot 2^{-2x^2} + 4} \leq 3 \end{aligned}$$

Therefore the two sides are equal if and only if $x = 0$.

Answer: $x = 0$.

33. Solve the inequality $\cos^2(x+1) \cdot \log(9 - 2x - x^2) \geq 1$.

Solution: The left side of the inequality is a product of two functions, $f(x) = \cos^2(x+1)$, $g(x) = \log(9 - 2x - x^2)$, and can be written as $f(x) \cdot g(x) \geq 1$. Because cosine is a bounded function, we know that $0 \leq \cos^2(x+1) \leq 1$. On the other hand, a logarithmic function with base 10 is a monotonically increasing function (for all x, such that $9 - 2x - x^2 > 0$). Thus the following must be true:

$$\begin{cases} f(x) \le 1 \\ g(x) \ge 1 \end{cases} \Rightarrow \begin{cases} \cos^2(x+1) \le 1 \\ \log(9-2x-x^2) \ge 1 \end{cases}$$

$$9 - 2x - x^2 \ge 10 \Leftrightarrow (x+1)^2 \le 0 \Leftrightarrow x = -1.$$

Answer: $x = -1$.

34. Solve the inequality $2^{\cos x} + 2^{\sin x} \ge 2^{1-\frac{\sqrt{2}}{2}}$.

Solution: Let us show that the left side is bounded below as

$$2^{\cos x} + 2^{\sin x} \ge 2\sqrt{2^{\cos x} \cdot 2^{\sin x}} = 2\sqrt{2^{\cos x + \sin x}} = 2\sqrt{2^{\sqrt{2}\cdot \sin\left(x+\frac{\pi}{4}\right)}} = 2^1 \cdot 2^{\frac{\sqrt{2}}{2}\sin\left(x+\frac{\pi}{4}\right)}$$

$$2^{\cos x} + 2^{\sin x} \ge 2^{1+\frac{\sqrt{2}}{2}\cdot \sin\left(x+\frac{\pi}{4}\right)} \ge 2^{1+\frac{\sqrt{2}}{2}}.$$

Because $2^{1+\frac{\sqrt{2}}{2}} > 2^{1-\frac{\sqrt{2}}{2}}$ is always true, then the given inequality is true for all real x.

You might want to learn about using an auxiliary argument in Chapter 3 before attempting this problem.

Answer: $x \in R$.

35. Solve the inequality $\log_{0.5}|1-x| - \log_{x-1}2 \le 2$.
Solution: First, we will find the domain of all functions: $\begin{cases} x-1>0 \\ x-1 \ne 1 \end{cases} \Leftrightarrow x \in (1,2) \cup (2,\infty)$.

Second, we will rewrite all logarithms with base 2:

$$\log_{x-1}2 = \frac{1}{\log_2(x-1)}$$

$$\log_{2^{-1}}|1-x| = \frac{\log_2|1-x|}{\log_2\frac{1}{2}} = -\log_2|1-x| = -\log_2(x-1)$$

Now the inequality can be rewritten as

$$-\log_2(x-1) - \frac{1}{\log_2(x-1)} \le 2$$

Considering its left side and using Property 4, Section 1.6, we will have the following two cases:
Case 1 If $\log_2(x-1) > 0$ so that $x > 2$, then

$$\left(\log_2(x-1) + \frac{1}{\log_2(x-1)}\right) \ge 2$$

or

$$-\left(\log_2(x-1)+\frac{1}{\log_2(x-1)}\right) \le -2 \le 2$$

The given inequality is true for all $x > 2$.
Case 2 If $\log_2(x-1) < 0$, using the domain of the logarithmic function, this is possible if $x \in (1,2)$ only and the following is true:

$$\log_2(x-1)+\frac{1}{\log_2(x-1)} \le -2$$
$$-\left(\log_2(x-1)+\frac{1}{\log_2(x-1)}\right) \ge 2$$

Then from this case, we can include only such values of x that make the last inequality an equality, i.e., $\log_2(x-1) = -1$. The solution of this equation is

$$x = 1+\frac{1}{2} = \frac{3}{2}.$$

Combining the two cases, we finally get the solution to this inequality as one value (3/2) and an interval.

Answer: $x > 3/2$ and $x > 2$.

36. (Budak) Solve the inequality: $\frac{x^2-1}{x^2+1} + x^2 - 5x + 6 < 0$.

Hint: Show that the function on the left-hand side is always positive.

Solution: We can rewrite the left side as

$$h(x) = f(x) + g(x),\ f(x) = \frac{x^2-1}{x^2+1},\ g(x) = x^2 - 5x + 6.$$

Also it is convenient to rewrite the functions as follows:

$$f(x) = \frac{x^2+1-2}{x^2+1} = 1 - \frac{2}{x^2+1}$$
$$g(x) = (x-2)(x-3) = \left(x-\frac{5}{2}\right)^2 - \frac{1}{4}$$

It helps us to see the boundedness of the functions and the intervals on which the functions are negative, positive, or zero. Thus, $f(x)$ is bounded and it has the upper bound, -1. The sign of $f(x)$ depends on the numerator, so

$$f(x) < 0,\ x \in (-1,1)$$
$$f(x) > 0,\ x \in (-\infty,\ -1) \cup (1,\infty)\,.$$
$$f(x) = 0,\ x = -1,\ x = 1$$

Similarly, $g(x)$ is also bounded and has the lower bound, $-1/4$. Also,

$$g(x) < 0,\ x \in (2,3)$$
$$g(x) > 0,\ x \in (-\infty,2) \cup (3,\infty)$$
$$g(x) = 0,\ x = 2,\ x = 3$$

Combining the behavior of the two functions, the following is true:

1. If $x \in (2,3)$, then

$$\begin{cases} -\dfrac{1}{4} \le g(x) < 0 \\ f(x) > 1 - \dfrac{2}{2^2+1} = \dfrac{3}{5} \end{cases} \Rightarrow h(x) = f(x) + g(x) > \frac{3}{5} - \frac{1}{4} = \frac{7}{20} > 0.$$

   There is no solution on this interval.
2. If $x \in \big(-\infty, 1\big] \cup [1,2] \cup (3,\infty)$, then $f(x) \ge 0,\ g(x) \ge 0 \Rightarrow h(x) \ge 0$. No solutions.
3. If $x \in (-1,1) \Rightarrow -1 < f(x) < 0,\ g(x) > 2 \Rightarrow h(x) > 1$. No solution.

Therefore, the function on the left is always positive!

Answer: No solution.

37. Solve the equation $\ln 3x + \frac{1}{\ln 3x} = 2\sin\left(3x + \frac{\pi}{6}\right)$.

Solution: Because the following is true:

$$\begin{cases} \left|\ln 3x + \dfrac{1}{\ln 3x}\right| \ge 2 \\ \left|2\sin\left(3x + \dfrac{\pi}{6}\right)\right| \le 2 \end{cases}$$

then applying Property 8, we can consider two cases:

Case 1 $\begin{cases} \ln 3x + \dfrac{1}{\ln 3x} = 2 \\ 2\sin\left(3x + \dfrac{\pi}{6}\right) = 2 \end{cases}$

or

Case 2 $\begin{cases} \ln 3x + \dfrac{1}{\ln 3x} = -2 \\ 2\sin\left(3x + \dfrac{\pi}{6}\right) = -2 \end{cases}$

Neither case has a solution.

Answer: No solutions.

38. Find all functions $f(x)$ defined $\forall x \in R$ and that $\forall x, y \in R$ satisfies the inequality: $|f(x+y) + \sin x + \sin y| < 2$.

Solution: If the given inequality is true for any real x and y, then we can check two simple cases:

a. Let $x = \frac{\pi}{2},\ y = \frac{\pi}{2}$ then $|f(\pi) + 2| < 2 \Rightarrow f(\pi) < 0$.
b. Let $x = \frac{3\pi}{2},\ y = -\frac{\pi}{2}$ then $|f(\pi) - 2| < 2 \Rightarrow f(\pi) > 0$.

We obtain a contradiction.
Therefore, $f(x)$ does not exist.

39. Solve the equation $\left(\sqrt{2-\sqrt{3}}\right)^x + \left(\sqrt{2+\sqrt{3}}\right)^x = 4$.

Hint: Use the fact that $\left(\sqrt{2-\sqrt{3}}\right)^x = \left(\frac{1}{\sqrt{2+\sqrt{3}}}\right)^x = y$, then the equation can be written as $y + \frac{1}{y} = 4$ and solved as the quadratic $y^2 - 4y + 1 = 0$, so $y_{1,2} = 2 \pm \sqrt{3}$.

Answer: $x = 2$.

40. Function $f(x)$ satisfies the following relationships for any real x and y: $f(x+y) = f(x) + f(y) + 80xy$. Evaluate $f\left(\frac{4}{5}\right)$ if $f\left(\frac{1}{4}\right) = 2$.
Solution:

$$\begin{aligned}
f\left(\frac{1}{2}\right) &= f\left(\frac{1}{4}+\frac{1}{4}\right) = 2 + 2 + 80\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = 9 \\
f(2x) &= 2f(x) + 80x^2 \\
f(1) &= f\left(\frac{1}{2}+\frac{1}{2}\right) = 9 + 9 + 80\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 38 \\
f(3x) &= 3f(x) + 3(80)x^2 \\
f(2) &= f(1+1) = 38 + 38 + 80(1)(1) = 156 \\
f(4x) &= 4f(x) + 6 \cdot 80 \cdot x^2 \\
f(4) &= f(2+2) = 156 + 156 + 80 \cdot 2 \cdot 2 = 682 \\
f(5x) &= 5f(x) + 10 \cdot 80 \cdot x^2
\end{aligned}$$

Now,

$$\begin{aligned}
f\left(5 \cdot \left(\frac{4}{5}\right)\right) &= 5f\left(\frac{4}{5}\right) + 10 \cdot 80 \cdot \left(\frac{4}{5}\right)^2 \Rightarrow \\
f(4) &= 5 \cdot f\left(\frac{4}{5}\right) + 512 \Rightarrow \\
632 &= 5 \cdot f\left(\frac{4}{5}\right) + 512
\end{aligned}$$

Solving for $f\left(\frac{4}{5}\right) = \frac{632-512}{5} = 24$, we obtain the answer.

Answer: $f\left(\frac{4}{5}\right) = 24$.

Chapter 2
Polynomials

This chapter covers problems involving polynomials. You will learn or review very important theorems, most with proofs, properties of polynomial functions starting from quadratic functions and ending with polynomial functions of the nth order. You will see the generalized form of the Vieta's Theorem and its special cases for quadratic, cubic, and quartic equations. Newton's Binomial Theorem will be introduced as will some well-known special products. In order to solve complex problems, we will need to review simple methods of solving polynomial equations of special types: then you will be ready to recognize when it might be helpful to make a special substitution that would simplify the equation, or the methods that can be used for an equation of a special type (for example, one that can be applied to a recurrent polynomial equation). We will spend some time working on the derivation of Cardano's formula for a general cubic equation and even demonstrate how Babylonians solved cubic equations of a special type. Though many facts can be found now on the Internet and using other sources, we will learn how to derive such formulas using the Vieta's Theorem or the Fundamental Theorem of Algebra, and also how to solve quartic equation by Ferrari or Euler methods. An important part of this chapter is the many problems with a parameter, and the variety of different approaches to solving such problems. Additionally, there are many problems involving integer solutions. For example, we know that a prime number in form $4m+1$ can be written as the sum of two squares, such as $2^2+3^2=13;\ \ 4^2+5^2=41$. Can you prove that for any quadratic equation $x^2+ax+1=b$ with natural roots, the expression a^2+b^2 is never a prime number? This and many other interesting and challenging problems will be solved in this chapter. Some topics will be familiar to you and some not. For example, we will solve polynomial equations in real variable and will look mainly for real solutions. However, we will demonstrate how sometimes it is convenient to evaluate a polynomial at an imaginary number in order to solve the equation in real numbers! This technique can be applied to the problem mentioned above and many others. Moreover, we will use Rolle's and Lagrange's Theorems and their

E. Grigorieva, *Methods of Solving Nonstandard Problems*,
DOI 10.1007/978-3-319-19887-3_2

applications by going one step beyond elementary mathematics content, in order to get additional proofs for our statements, or in order to see how different fields of mathematics are connected.

2.1 Introduction to Polynomial Equations: Important Theorems

In general, a polynomial of nth degree in one variable x can be written as $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0, \quad a_n \neq 0.$

In this book, we consider only polynomials with real and most of the time with integer coefficients.

Theorem 10 *Two polynomials in x are equal if and only if they have the same degree and equal coefficients for corresponding powers of x.*

Theorem 11 *If a product of two polynomials equals zero, then at least one of the polynomials equals zero.*

For example, if $(x-1)(x+3)=0$, then $x=1$ or $x=-3$.

Theorem 12 *For any two polynomials $P(x)$ and $D(x)$, $(D(x) \neq 0)$ there exists a unique pair of polynomials $q(x)$ and $r(x)$ such that $P(x) = D(x)q(x) + r(x)$.*

This theorem can also be written in the following form:

$$\frac{P(x)}{D(x)} = q(x) + \frac{r(x)}{D(x)}.$$

From which we can see that the degree of the remainder polynomial $r(x)$ must be less than the degree of the divisor polynomial $D(x)$. Thus if we divide a polynomial of the third degree by a quadratic polynomial, then the remainder's degree must be less than 2. Therefore, the remainder polynomial will be either linear or a constant.

In order to understand it, we can think of division with and without a remainder. For example, 15 is divisible by 3(divisor) and can be written as $15 = 3 \cdot 5 + 0$. 16 is not divisible by 3 and when divided by 3 gives a remainder of 1. Thus $16 = 3 \cdot 5 + 1$ or it can be written as $\frac{16}{3} = 5 + \frac{1}{3}$. Polynomials also can be divided with a remainder

or without a remainder. For example, cubic polynomial $P(x) = x^3 - 2x^2 - x - 2$ is divisible by $D(x) = x^2 - 1$ with the resulting quotient $q(x) = x - 2$: so we can write $x^3 - 2x^2 - x - 2 = (x^2 - 1) \cdot (x - 2)$.

On the other hand, $p(x) = x^3 - 2x^2 - 2$ is not divisible by $D(x) = x^2 - 1$, and after long division can be written as $x^3 - 2x^2 - 2 = (x^2 - 1) \cdot (x - 2) + [x - 4]$. Therefore, $[x - 4]$ is the remainder. Division with a remainder can also be written in the equivalent form as

$$\frac{x^3 - 2x^2 - 2}{x^2 - 1} = x - 2 + \frac{x - 4}{x^2 - 1}.$$

Theorem 13 *(**Bezout's**) The remainder from the division of polynomial $P(x)$ by the binomial $(x-a)$ equals the value of the polynomial $P(x)$ at $x=a$, i.e., $r(a)=P(a)$.*

Proof Consider the equality $P_n(x) = (x - a)q(x) + r(x)$. Substituting a for x we obtain $P_n(a) = (a - a)q(a) + r(a) = r(a)$.

Therefore, we state the following.

Theorem 14 *Polynomial $P(x)$ is divisible by the binomial $(x-a)$ if and only if the value of the polynomial at $x = a$ equals zero, i.e., $P(a)=0$.*

We can also say that if $x = a$ is not zero of a polynomial, then $P(a) \neq 0$.

This theorem helps us quickly create a polynomial of any desired degree with a given zero. For example, let $x = 1$ be a zero; then we can claim that the polynomial $7x^4 - 5x^3 - x^2 + 3x - 4 = 0$ has this root because simple arithmetic operation gives us $7 \cdot 1^4 - 5 \cdot 1^3 - 1^2 + 3 \cdot 1 - 4 = 0$. This simple trick can be used in a classroom when a polynomial with certain zeroes must be created for demonstration. You can try any integer as a zero, but the simplest way is to always try $x=1$.

Let us see how this property can be used for the following problem:

Problem 36 Let a, b, c be real numbers such that no pair of these numbers is equal. Prove the expression $a^2(c - b) + b^2(a - c) + c^2(b - a) \neq 0$.

Proof If no pair of the parameters is equal, then no difference of two parameters is zero, i.e., $a - b \neq 0$ or $b - c \neq 0$ or $a - c \neq 0$. However, it looks like the given expression is divisible by $(a - b)(b - c)(a - c)$.

Let us prove this as follows:

$$a^2(c-b)+b^2(a-c)+c^2(b-a)=a^2c-a^2b+b^2a-b^2c+c^2b-c^2a$$
$$a^2(c-b)-a(c-b)(c+b)+bc(c-b)=(c-b)(a^2-ac-ab+bc)$$
$$=(c-b)(a-b)(a-c).$$

There are several corollaries of Bezout's Theorem.

Corollary 1 *Polynomial* $P_n(x)=x^n-a^n$ *is divisible by binomial* $(x-a)$ *for any natural* n.

Proof Obviously, $P_n(a)=a^n-a^n=0$.

Corollary 2 *Polynomial* $P_n(x)=x^n-a^n$ *is divisible without remainder by binomial* $(x+a)$ *for any even degree* n $(n=2k,\ k\in N)$.

Proof In fact, $P_{2k}(-a)=(-a)^{2k}-a^{2k}=0$.

Corollary 3 *Polynomial* $P_n(x)=x^n+a^n$ *is divisible without remainder by binomial* $(x+a)$ *for any odd degree* n $(n=2k+1,\ k\in N)$.

Proof In fact, $P_{2k+1}(-a)=(-a)^{2k+1}+a^{2k+1}=0$.

The following problem will demonstrate application of these corollaries and Bezout's Theorem.

Problem 37 Prove that for any even n the number $(20^n+16^n-3^n-1)$ is divisible by 19.

Proof Because $n=2k, k\in N$, then using special product formulas (see more about special product formulas in the following section) we obtain

$$20^n+16^n-3^n-1=\left(20^{2k}-1\right)+\left(16^{2k}-3^{2k}\right)$$
$$=\left(20^k-1\right)\left(20^k+1\right)+\left(16^k+3^k\right)\left(16^k-3^k\right)$$

In the first term, the first factor $\left(20^k-1\right)$ is divisible by 19 for any value of k.

In the second term, if $k=2m+1$ then the second factor of it can be written as $(16^m+3^m)(16^m-3^m)$. Thus if m is odd, then decomposition of the second factor (16^m-3^m) is completed. If m is even, then we continue to factor; after a finite

number of steps, that is at most $(n-1)$, the factorization will be completed and one of the factors of this decomposition will have the form of (16^s+3^s), $s=2l+1$ (s is an odd number). Then this factor (16^s+3^s) is divisible by 19. We showed that both terms above are divisible by 19 $\forall k \in N$; therefore, $20^n+16^n-3^n-1$ is divisible by 19 for any even natural number n.

Definition A number a is a zero of polynomial $P(x)$ if $P(a)=0$.

Theorem 15 *A number a is a zero of polynomial $P(x)$ if and only if the polynomial is divisible by $(x-a)$.*

Theorem 16 *(Rational Zero Theorem). If $P(x)$ is a polynomial with integer coefficients and if $\frac{p}{q}$ is a zero of $P(x)$, $\left(P\left(\frac{p}{q}\right)=0\right)$, then p is a factor of the constant term a_0 of $P(x)$ and q is a factor of the leading coefficient a_n of $P(x)$ $(a_0 \neq 0,\quad a_n \neq 0)$.*

We can use the Rational Zero Theorem to find all the rational zeroes of a polynomial. Here are the steps:

1. Arrange the polynomial in descending order.
2. Write down all the factors of the constant term. These are all the possible values of p.
3. Write down all the factors of the leading coefficient. These are all the possible values of q.
4. Write down all the possible values of $\frac{p}{q}$. Remember that since factors can be negative, $\frac{p}{q}$ and $-\frac{p}{q}$ must both be included. Simplify each value and cross out any duplicates.
5. Use synthetic or long-term division to determine the values of $\frac{p}{q}$ for which $P\left(\frac{p}{q}\right)=0$. These are all the rational roots of $P(x)$.

Example Find all rational zeroes of $P(x)=x^3-9x+9+2x^4-19x^2$.

1. $P(x)=2x^4+x^3-19x^2-9x+9$.
2. Factors of the constant term: ±1, ±3, ±9.
3. Factors of the leading coefficient: ±1, ±2.

Figure 2.1 Synthetic division

1	2	1	-19	-9	9
		2	3	-16	-25
	2	3	-16	-25	-16

Remainder = -16. Not a zero.

-1	2	1	-19	-9	9
		-2	1	18	-9
	2	-1	-18	9	0

Remainder = 0. Is a zero.

1/2	2	1	-19	-9	9
		1	1	-9	-9
	2	2	-18	-18	0

Remainder = 0. Is a zero.

-1/2	2	1	-19	-9	9
		-1	0	19/2	-1/4
	2	0	-19	1/2	35/4

Remainder = 35/4. Not a zero.

3	2	1	-19	-9	9
		6	21	6	-9
	2	7	2	-3	0

etc. Remainder = 0. Is a zero.

4. Possible values of $\frac{p}{q}: \pm\frac{1}{1}, \pm\frac{1}{2}. \pm\frac{3}{1}; \pm\frac{3}{2}, \pm\frac{9}{1}, \pm\frac{9}{2}$: which can be simplified to: $\frac{p}{q}: \pm 1, \pm\frac{1}{2}. \pm 3; \pm\frac{3}{2}, \pm 9, \pm\frac{9}{2}$.
5. Use synthetic division (Figure 2.1).

Remark If $x = a$ is the root of polynomial $p_n(x)$ and $x = b$ is also the root of $p_n(x)$, then $p_n(x) = (x - a)q_{n-1}(x)$ and $(x - b)$ must be a factor of q_{n-1}, such that $q_{n-1} = (x - b)s_{n-2}(x)$, etc. Therefore as soon as we find one of the rational zeroes, for example, when we find $p_n(a) = 0$ we can continue to divide the resulting polynomial by the next zero until we are done. For example, because in the previous example $x = -1$, $x = 3$, $x = -3$ are zeroes, then we could obtain the following by using Horner synthetic division:

$$p_n(x) = (x+1)(x-3)(x+3)(2x-1) = 0$$

Hence the last root is $x = 1/2$.

Theorem 17 *If all coefficients of a polynomial of nth degree are integers ($n \geq 1$) and a zero of this polynomial is also an integer, then it is a factor of the constant term.*

Proof Consider a polynomial $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0, \quad n \geq 1, \quad a_n \neq 0$, with a zero $x = a$. Suppose that we can divide $p(x)$ by the binomial $(x-a)$ with a remainder, and obtain the divisor as a polynomial of degree $(n-1)$, i.e., $q_{n-1}(x) = b_{n-1}x^{n-1} + \ldots + b_3x^3 + b_2x^2 + b_1x + b_0, \quad n \geq 1$, $b_{n-1} \neq 0$ and the remainder r.

Because all coefficients of $p(x)$ are integers and number a is also integer, then the numbers $b_{n-1}, \ldots b_3, b_2, b_1, b_0$ and r are also integers.

Using synthetic division (**William George Horner**, British mathematician, 1786–1837, functional analysis, number theory, known for Horner synthetic division algorithm) we obtain that $r = a_n + a \cdot b_{n-1}$. However, it follows from Theorem 13 that if a is a root of a polynomial then $r(a) = 0$; hence $a_n = a \cdot (-b_{n-1})$.

Since a_n, a and $(-b_{n-1})$ are integer numbers, a must be a factor of a_n. The theorem is proven.

Corollary 4 *Integer zeroes of a polynomial are the factors of its constant term.*

Problem 38 Does polynomial $p_4(x) = x^4 + 2x^3 - 2x^2 - 6x + 5$ have any integer roots?

Solution Divisors of a constant term are: 1, −1, 5, and −5.

Let us evaluate the polynomial at each of the possible integer roots:

$$p_4(1) = 1^4 + 2 \cdot 1^3 - 2 \cdot 1^2 - 6 \cdot 1 + 5 = 0 \Rightarrow x = 1 \text{ is a zero}$$
$$p_4(-1) = (-1)^4 + 2 \cdot (-1)^3 - 2 \cdot (-1)^2 - 6 \cdot (-1) + 5 = 8 \neq 0$$
$$\Rightarrow x = -1 \text{ is not a zero}$$
$$p_4(5) = 1^4 + 2 \cdot 5^3 - 2 \cdot 5^2 - 6 \cdot 5 + 5 = 800 \neq 0 \Rightarrow x = 5 \text{ is not a zero}$$
$$p_4(-5) = (-5)^4 + 2 \cdot (-5)^3 - 2 \cdot (-5)^2 - 6 \cdot (-5) + 5 = 360 \neq 0$$
$$\Rightarrow x = -5 \text{ is not a zero}$$

Therefore, only integer $x = 1$ makes the polynomial zero. However, a polynomial of 4th degree can have at most 4 real roots; furthermore, some roots can have multiplicity more than one. Applying Horner's algorithm (synthetic division) let us divide $p(x)$ by $(x-1)$:

$$p_4(x) = x^4 + 2x^3 - 2x^2 - 6x + 5 = (x-1)\left(x^3 + 3x^2 + x - 5\right).$$

Next, we will consider polynomial $p_3(x) = x^3 + 3x^2 + x - 5$. There is no need to evaluate this at $x = -1$, 5, and -5. However, let us evaluate it at $x = 1$:

$$p_3(x) = 1^3 + 3 \cdot 1^2 + 1 - 5 = 0 \Rightarrow x = 1 \text{ is zero.}$$

Applying **Horner's** algorithm again we finally obtain

$$p_4(x) = x^4 + 2x^3 - 2x^2 - 6x + 5 = (x-1)^2\left(x^2 + 4x + 5\right).$$

The second factor does not have real zeroes (its discriminant is negative, see more in the following section). The given polynomial has two integer roots $x = 1$ and $x = 1$ or we say that it has an integer root $x = 1$ of multiplicity 2.

Definition If $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0$, $n \geq 1$, $a_n \neq 0$ is divisible by $(x-a)^k$ then $x = a$ is zero of the polynomial of multiplicity k.

Theorem 18 If a polynomial $p_n(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0$, $n \geq 1$ with integer coefficients and with the leading coefficient equal to 1 has a rational zero, then that zero is an integer.

Proof We will prove this by contradiction. Assume that a zero of the polynomial can be written as $a = \frac{p}{q}$ where p and q are relatively prime. Then the following must be true:

$$\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + a_3\left(\frac{p}{q}\right)^3 + a_2\left(\frac{p}{q}\right)^2 + a_1\frac{p}{q} + a_0 = 0,$$

which can be written as

$$\frac{p^n}{q^n} = -\left(a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + a_3\left(\frac{p}{q}\right)^3 + a_2\left(\frac{p}{q}\right)^2 + a_1\frac{p}{q} + a_0\right)$$

Multiplying this by q^{n-1} we will obtain an equivalent equality

$$\frac{p^n}{q} = -a_{n-1}p^{n-1} - a_{n-2}p^{n-2}q - \ldots - a_1 pq^{n-2} - a_0 q^{n-1}.$$

Because p and q are relatively prime, then $\frac{p^n}{q}$ cannot be an integer number. On the other hand, the right side is an integer number. Such equality cannot occur. Therefore, our assumption was wrong. The theorem is proven.

Problem 39 Find all real zeroes of $x^3 + 2x^2 - 3136 = 0$.

Solution By the Rational Zero Theorem if this polynomial has a rational zero, then it is an integer. Hence we will factorize 3136:

$$3136 = 2 \cdot 1568 = 2 \cdot 2 \cdot 784 = 2 \cdot 2 \cdot 2 \cdot 392 = 2^6 \cdot 7^2$$

By substitution we can see that $x = 14$ is a zero, then $(x - 14)$ is a factor of $p(x) = x^3 + 2x^2 - 3136 = 0$.

Hence, we can divide the given polynomial by $(x - 14)$ using long division or Horner synthetic division and obtain

$$x^3 + 2x^2 - 3136 = (x - 14)\left(x^2 + 16x + 224\right) = 0$$

The second factor never equals zero over the set of real numbers. You can check it yourself by finding that the discriminant of the quadratic equation is negative (-160).

Answer $x = 14$.

Remark There is another proof that $x = 14$ is the only real solution for the problem above. If we take a derivative of $p(x)$, we obtain, $p'(x) = 3x^2 + 4x = x \cdot (3x + 4) = 0$. Then the polynomial function will increase monotonically for $x > 0$ and $x < -4/3$ and it will decrease on $(-4/3, 0)$. At $x = 0$ $p(0) = -3136$, at $x = 14$ $p(14) = 0$; therefore this zero $x = 14$ is unique. This problem is also solved later in this book using the Babylonian's approach.

Corollary 5 *For any polynomial with integer coefficients and unit leading coefficient, all its rational zeroes are integers.*

Let us see how it can be applied to the problem below.

Problem 40 Find all real solutions of the equation $x^5 + x - 34 = 0$.

Solution The function on the left is monotonically increasing over the entire set of real numbers because $f(x) = f_1(x) + f_2(x)$, where both $f_1(x) = x^5, f_2(x) = x - 34$ are monotonically increasing functions. Therefore, if we find a divisor of 34 that would make the function zero, then that value of x is the only real root that the function has. Divisors of 34 are $\pm1; \pm2; \pm17; \pm34$ and $x = 2$ is the zero because $f(2) = 0$.

Answer $x = 2$.

You can practice more on such functions in the HW section.

Consider a polynomial $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0$, $n \geq 1$ with integer coefficients and another polynomial

$$\begin{aligned}Q_n(x) &= {a_n}^{n-1}P_n(x)\\ &= (a_nx)^n + a_{n-1}(a_nx)^{n-1} + \ldots\\ &\quad + a_3{a_n}^{n-4}(a_nx)^3 + a_2{a_n}^{n-3}(a_nx)^2 + a_1{a_n}^{n-2}(a_nx) + {a_n}^{n-1}a_0\end{aligned}$$

It is obvious that $P_n(x)$ and $Q_n(x)$ have the same zeroes. Denote $y = a_nx$, then $P_n(x) \; = T_n(x)$:

$$T_n(x) = y^n + a_{n-1}y^{n-1} + \ldots + a_3{a_n}^{n-4}y^3 + a_2{a_n}^{n-3}y^2 + a_1{a_n}^{n-2}y + {a_n}^{n-1}a_0$$

By Theorem 18 this polynomial has only integer zeroes $y_1, y_2, \ldots y_m$, then the numbers $x_k = \frac{y_k}{a_n}, \quad k \in \{1, 2, 3, \ldots, m\}$.

And they will be the only rational zeroes of $Q_n(x)$. Thus for any polynomial with integer coefficients we can always find all its rational zeroes. If a polynomial has rational coefficients we can always multiply it by the greatest common denominator and then find solutions of the corresponding polynomial.

Problem 41 Find real solutions of the following equation:

$$P(x) = x^3 + \frac{1}{2} \cdot x^2 - \frac{1}{4} \cdot x - \frac{1}{8} = 0.$$

Solution Multiplying by 8 we will obtain the equivalent polynomial equation $8x^3 + 4x^2 - 2x - 1 = 0$, which if $y = 2x$ can be written as

$$T(y) = y^3 + y^2 - y - 1 = 0.$$

The only rational roots of this equation must be integers and factors of (-1).

$T(1) = T(-1) = 0$, then $(y+1)(y-1)$ are factors of the polynomial $T(y)$. Applying Horner's algorithm we obtain

$$y^3 + y^2 - y - 1 = (y-1)(y+1)(y+1) = 0$$

Therefore $T(y)$ has three zeroes, 1, -1, and -1, and hence $P(x)$'s zeroes are 1/2, $-1/2$, and $-1/2$.

Remark This method was known to ancient Babylonians. See more problems later in this chapter.

Any polynomial of nth degree has n zeroes. However, it has at most n real zeroes. Some zeroes are not necessarily real.

Theorem 19 *(Fundamental Theorem of Algebra). Any polynomial of nth degree has n complex zeroes.*

Moreover, if $x = a + ib$ is a complex zero of a polynomial then its complex conjugate $a - ib$ is also the root of the function.

Simple behavior of the graphs of polynomial functions predicted based on their degree and leading coefficients can be summarized as follows:

- The graph of a polynomial function of even degree with positive leading coefficient increases on the left and on the right.
- The graph of a polynomial function of even degree with negative leading coefficient decreases on the left and on the right.
- The graph of a polynomial function of odd degree with positive leading coefficient decreases on the left and increases on the right.
- The graph of a polynomial function of odd degree with negative leading coefficient increases on the left and decreases on the right.

Remark 1 It is clear that polynomials of odd degree will always have at least one X-intercept (real zero). Moreover, if $p_n(x) = f_1(x) + f_2(x), n = 2k+1, k \in N$ and if both $f_1(x), f_2(x)$ are monotonically increasing or decreasing functions, then the polynomial function has only one real zero. You can try to investigate the following functions: $f(x) = x^3 + 3x - 3, g(x) = 2x^3 + x - 18$ or solve similar problems in the HW.

Remark 2 We can also explain the "left" and "right" behavior of a polynomial function by the sign of the leading coefficient (the coefficient of the highest degree term) and by the degree (odd or even n). For example, for a polynomial function of nth degree $y = a_n x^n + p_{n-1}(x)$ as x increases or decreases without bound, the behavior of the function at the ends is dominated by the term of the largest degree, $a_n x^n$. Thus, for a positive leading coefficient, and odd degree, the function $y = a_n x^n$ is monotonically increasing and it falls to the left and rises to the right.

On the other hand, if the function $y = a_n x^n$ is of even degree and positive a_n, then it rises to both ends, as does the entire polynomial $y = a_n x^n + p_{n-1}(x)$.

Problem 42 Find a polynomial function that has the given zeroes and Y-intercept. Zeroes: $3i$, -2, 1 and the Y-intercept $(0, 18)$.

Solution

$$\begin{aligned} y &= a(x-3i)(x+3i)(x+2)(x-1) = a(x^2+9)(x^2+x-2) \\ &= a(x^4+x^3+7x^2+9x-18) \\ y(0) &= 18 \Rightarrow a = -1. \end{aligned}$$

Answer $y = -\left(x^4 + x^3 + 7x^2 + 9x - 18\right)$.

Theorem 20 ***(Vieta's Theorem)***. *If $x_1, x_2, x_3 \ldots x_n$ are the zeroes of the polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$, $(a_n \neq 0)$, then the following relationships are true:*

$$\begin{cases} x_1 + x_2 + \ldots + x_n = -\dfrac{a_{n-1}}{a_n} \\ x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n = \dfrac{a_{n-2}}{a_n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_1 \cdot x_2 \cdot \ldots \cdot x_n = (-1)^n \dfrac{a_0}{a_n} \end{cases}$$

Vieta's Theorem is very important and is most applicable to quadratic and cubic polynomials. For example, for a polynomial of fourth degree with the unit leading coefficient $p(x) = x^4 + ax^3 + bx^2 + cx + d = 0$ with four real roots, x_1, x_2, x_3, x_4 the following is valid:

$$\begin{cases} x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = b \\ x_1x_2x_3x_4 = d \\ x_1 + x_2 + x_3 + x_4 = -a \\ x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4 + x_1x_3x_4 = -c \end{cases}$$

Problem 43 Factor the polynomial $(x+y+z)^3 - x^3 - y^3 - z^3$.

Solution Note that if we substitute $x = -y$, $y = -z$ or $x = -z$, then the given expression becomes zero. Therefore, by Bezout's Theorem, the given expression must be divisible by $(x+y)(x+z)(y+z)$. Finally, the answer is $3(x+y)(x+z)(y+z)$. The coefficient 3 can be explained by the formula for the cube of a sum. Thus, the given expression is divisible by $(x+y)$, and we can rewrite it as

$$\begin{aligned}((x+y)+z)^3 - x^3 - y^3 - z^3 &= (x+y)^3 + 3(x+y)^2 z + 3(x+y)z^2 - (x^3+y^3) \\ &= 3(x+y)(z^2 + z(x+y) + xy) \\ &= 3(x+y)(z+x)(z+y).\end{aligned}$$

Two other factors come from Vieta's Theorem applied to a quadratic function in variable z.

Answer $3(x+y)(x+z)(y+z)$.

Sometimes, it is helpful to consider a polynomial at a complex number, i, $P(i)$; then $P(-i)$ will be its complex conjugate, and their product will be a real number. For example, consider a linear function $P(x) = ax + b$. Evaluate $P(i) = ai + b$, $P(-i) = -ai + b$, then $P(i) \cdot P(-i) = (b + ia)(b - ia) = b^2 + a^2 \in R$.

In the following problem we will apply these ideas; however, it could also be solved with the use of Vieta's Theorem. Please try it yourself.

Problem 44 Polynomial function $p(x) = x^3 + ax^2 + bx + c$ has three real roots. Assuming that $|c - a| = 8$, $b = 7$, evaluate $\left(x_1^2 + 1\right)\left(x_2^2 + 1\right)\left(x_3^2 + 1\right)$.

Solution If x_1, x_2, x_3 are the roots, then the polynomial can be factored as $p(x) = (x - x_1)(x - x_2)(x - x_3)$, and by the ideas mentioned above, we have

$$p(i) \cdot p(-i) = \left(1 + x_1^2\right)\left(1 + x_2^2\right)\left(1 + x_3^2\right)$$

On the other hand,

$$\begin{aligned}p(i) &= i^3 + ai^2 + bi + c = -i - a + bi + c = (c - a) + i \cdot (b - 1) \\ p(-i) &= (-i)^3 + a(-i)^2 + b(-i) + c = i - a - bi + c \quad = (c - a) - i \cdot (b - 1) \\ p(i) \cdot p(-i) &= (c - a)^2 + (b - 1)^2\end{aligned}$$

Therefore, $\left(1+x_1^2\right)\left(1+x_2^2\right)\left(1+x_3^2\right)=(c-a)^2+(b-1)^2=8^2+(7-1)^2=100.$

Answer $\left(1+x_1^2\right)\left(1+x_2^2\right)\left(1+x_3^2\right)=(c-a)^2+(b-1)^2=100.$

2.2 Quadratic Functions and Quadratic Equations

In order to solve many problems about polynomials, we need to review quadratic functions and quadratic equations. A function $f(x)=ax^2+bx+c,\ a\neq 0$ is called a quadratic function, and the graph of it is a parabola. Unfortunately, just a few facts are studied in high schools such as that a parabola opens upward if $a>0$ and downward if $a<0$, that all parabolas are symmetric with respect to the line $x=x_v=-\dfrac{b}{2a}$, and that the point $(x_v, f(x_v))$ is the vertex of the parabola.

When a student is asked to find zeroes of a quadratic function or solve the corresponding quadratic equation

$$ax^2+bx+c=0 \tag{2.1}$$

he or she usually factors the equation using the FOIL technique or tries to use graphing technology.

However, it is important to introduce the **discriminant** (here and below we denote it by $\boldsymbol{D}$) of a quadratic equation: a special feature of each quadratic equation that determines the number of its real zeroes or X-intercepts of the quadratic function, respectively.

The roots of a quadratic equation $ax^2+bx+c=0$ can be found as

$$x_{1,2}=\frac{-b\pm\sqrt{D}}{2a},\quad D=b^2-4ac \tag{2.2}$$

If $D\geq 0$, then a quadratic equation has two real roots, and if $D<0$, then it has no real roots.

Proof Let us factor $\boldsymbol{a}$ out and complete the square inside parentheses:

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}\cdot x+\frac{c}{a}\right)=0$$

$$a\left(x^2+2\frac{b}{2a}\cdot x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)=0$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Denote $D = b^2 - 4ac$ and substitute it into the previous equation:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{D}{4a^2}$$

If $D > 0$, then the right side of the equation above is positive and it can be solved for x as

$$x + \frac{b}{2a} = \frac{\sqrt{D}}{2a} \quad \text{or } x + \frac{b}{2a} = -\frac{\sqrt{D}}{2a}$$

Both roots can be written as

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$$

If $D > 0$, then a quadratic equation has two real roots and the quadratic function intersects the X-axis at two points.

If $D = 0$ then we obtain one real root $x = -\frac{b}{2a}$ of multiplicity two. Hence, the quadratic function has only one X-intercept.

If $D < 0$, then (2.1) has no real roots and the quadratic function does not intersect the X-axis!

If the coefficient b of the linear term of a quadratic equation is an even, then it is better to use the so-called **D/4 formula**. Dividing all terms of (2.2) by 2 we obtain

$$x_{1,2} = \frac{-\frac{b}{2} \pm \sqrt{\frac{b^2 - 4ac}{4}}}{2 \cdot \frac{a}{2}} = \frac{-\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - ac}}{a}$$

Thus D/4 formula can also be written as

$$x_{1,2} = \frac{-\frac{b}{2} \pm \sqrt{\frac{D}{4}}}{a}, \quad \text{where } \frac{D}{4} = \left(\frac{b}{2}\right)^2 - ac \tag{2.3}$$

The advantage of using this formula can be demonstrated by the following example.

Solve the equation

$$7x^2 - 4x - 1 = 0$$

$$\frac{D}{4} = 2^2 + 7 = 11$$

$$x_{1,2} = \frac{2 \pm \sqrt{11}}{7}$$

Additionally, it gives us a solution in the simplest form.

Problem 45 For what value of a parameter a does the quadratic function $f(x) = (5a - 1)x^2 - (5a + 2)x + 3a - 2$ have one X-intercept?

Solution If the discriminant equals zero then there is one X-intercept:

$$D = (5a + 2)^2 - 4(5a - 1)(3a - 2) = 0$$
$$35a^2 - 72a + 4 = 0$$
$$a = 2, a = \frac{2}{35}.$$

Answer $a = 2, a = \frac{2}{35}$.

2.2.1 Vieta's Theorem for a Quadratic Equation

The French mathematician Vieta is well known for his work in number theory and analysis. Rational and of course integer zeroes x_1, x_2 of a quadratic equation $ax^2 + bx + c = 0, \ a \neq 0$ can be found using Vieta's Theorem.

Vieta's Theorem: If x_1, x_2 are the roots of a quadratic equation $ax^2 + bx + c = 0, \ a \neq 0$, then

$$x_1 \cdot x_2 = \frac{c}{a}$$
$$x_1 + x_2 = -\frac{b}{a}$$

Reversed Vieta's Theorem: If numbers x_1, x_2 satisfy the following system: $\begin{cases} x_1 \cdot x_2 = q \\ x_1 + x_2 = -p \end{cases}$ then x_1, x_2 are the roots of a quadratic equation

$$x^2 + px + q = 0.$$

Though Vieta's Theorem is not well known in the USA, some students can factor quadratic trinomials mentally using FOIL. If you have such skills go ahead and factor, but do not forget to check your factorization by multiplication. Of course, every technique has an underlying explanation. Let us expand the expression

$$(x+\alpha)(x+\beta)=x^2+(\alpha+\beta)x+\alpha\beta=0$$

Working backward this formula is FOIL.

On the other hand, if we need to solve $x^2+(\alpha+\beta)x+\alpha\beta=0$ instead, we could apply Vieta's Theorem and obtain

$$\begin{cases} x_1\cdot x_2=\alpha\beta \\ x_1+x_2=-(\alpha+\beta) \end{cases} \Rightarrow x_1=-\alpha, x_2=-\beta.$$

This gives the same zeros.

Vieta's Theorem and its reverse are very useful, especially when you can guess one of the zeroes or that zero is given.

Example You can see by substitution that $x=1$ is the root of

$$17x^2+4x-21=0$$

Since $a=17$, $b=4$, $c=-21$, and $x_1=1$, then from the first formula of Vieta's Theorem we obtain right away that the second root is $x_2=-\frac{21}{17}$.

Remark When I teach lower level math classes and I have to create a quadratic equation with real roots, I quickly in my mind make one of the roots 1 and then by "playing" with the coefficients, I can make several equations with this root, by mentally satisfying the equation

$a+b+c=0$. For example, I can set $a=12$, $b=-5$, then $c=-7$, and $x_2=-\frac{7}{12}$.

Or, if I chose $a=4$, $b=-15$, then $c=9$ and $x_2=\frac{9}{4}$, etc. It is a very useful trick when you do not have a calculator and have to check your students' work.

If $a=1$ the factoring looks easier:

$$\begin{aligned} &x^2+bx+c=(x-x_1)(x-x_2), \quad \text{where} \\ &x_1\cdot x_2=c \\ &x_1+x_2=-b \end{aligned}$$

Therefore, we are looking for such numbers x_1 and x_2, the product of which equals the constant term (c), and the sum of which adds up to a negative coefficient of x.

Example $x^2-7x+6=(x-1)(x-6)$ because

$$\begin{aligned} &1\cdot 6=6 \\ &1+6=-(-7)=7 \end{aligned}$$

Let us apply Vieta's Theorem to the following problems.

Problem 46 It is known that x_1, x_2 are the roots of the equation $2x^2 - \left(\sqrt{3}+5\right)x - \sqrt{4+2\sqrt{3}} = 0$. Find the value of $A = x_1 + x_1 \cdot x_2 + x_2$.

Solution Expression A contains the sum and the product of the zeroes of the quadratic equation, so it will be useful to apply Vieta's Theorem right away. We obtain the following: $A = (x_1 + x_2) + x_1 \cdot x_2 = \frac{\sqrt{3}+5}{2} - \frac{\sqrt{4+2\sqrt{3}}}{2}$.

This can be the answer but it does not look nice and we will try to simplify it. Consider the radicand of the second term, $4 + 2\sqrt{3}$, and try to recognize in it a trinomial square. Indeed, $4 + 2\sqrt{3} = 1 + 2\sqrt{3} + \left(\sqrt{3}\right)^2 = \left(\sqrt{3}+1\right)^2$. Since $\sqrt{3} + 1 > 0$, then $\sqrt{4+2\sqrt{3}} = \sqrt{3} + 1$. $\left(\sqrt{a^2} = |a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}\right)$.

Finally, we can evaluate A as

$$A = \frac{\sqrt{3}+5}{2} - \frac{\sqrt{3}+1}{2} = \frac{\sqrt{3}+5-\sqrt{3}-1}{2} = 2.$$

Answer $A = 2$.

Problem 47 Find such parameter a, for which the sum of the squares of the roots of a quadratic equation $x^2 + (3a - 1)x + a = 0$ equals 1.

Solution Applying Vieta's Theorem we have the system below:

$$\begin{cases} x_1 + x_2 = 1 - 3a \\ x_1 \cdot x_2 = a \\ x^2{}_1 + x^2{}_2 = 1 \end{cases}$$

In order to have two roots, the discriminant must be positive:

$$D = (3a - 1)^2 - 4a = 9a^2 - 10a + 1 > 0$$

By completing the square the last equality can be rewritten in terms of the sum and product of the roots as

$$\begin{aligned} &(x_1 + x_2)^2 - 2 \cdot x_1 \cdot x_2 = 1 \\ &(3a - 1)^2 - 2 \cdot a = 1 \\ &9a^2 - 8a = 0 \\ &a = 0, \quad a = \frac{8}{9} \end{aligned}$$

Finally, substituting both values of the parameter into discriminant formula, we conclude that the quadratic equation has two real roots only at $a=0$.

Answer $a=0$.

Problem 48 x_1, x_2 are the roots of a quadratic equation $x^2+bx-1=0$. Evaluate $x_1^3+x_2^3$.

Solution Let us rewrite the requested expression using the formula of the cube of a sum:

$$(x_1+x_2)^3 = x_1^3+x_2^3+3x_1x_2\cdot(x_1+x_2)$$
$$x_1^3+x_2^3 = (x_1+x_2)^3-3x_1x_2\cdot(x_1+x_2)$$
$$x_1^3+x_2^3 = (-b)^3-3(-b)(-1) = -b^3-3b$$

In the last expression we substituted the sum and the product of the roots given by Vieta's Theorem.

Answer $x_1^3+x_2^3=-b^3-3b$.

Problem 49 For what value of a parameter a is the ratio of the roots of quadratic equation $x^2+ax-16=0$ equal to -4?

Solution We will apply Vieta's Theorem as follows:

$$\begin{cases} x_1\cdot x_2=-16 \\ \dfrac{x_1}{x_2}=-4 \end{cases} \Rightarrow \begin{cases} x_1=-4x_2 \\ -4(x_2)^2=-16 \end{cases} \Leftrightarrow \left[\begin{array}{l} x_2=2 \text{ or } x_2=-2 \\ x_1=-8 \text{ or } x_1=8 \end{array}\right.$$

Therefore, $a=-(x_1+x_2)=\pm 6$.

Answer $a=\pm 6$.

Problem 50 For what values of parameters p and q does the difference of the roots of $x^2+px+q=0$ equal 5 and the difference of their cubes equal 35?

Solution Let us use the following formula:

$$x_1^3 - x_2^3 = (x_1 - x_2)\left(x_1^2 + x_1x_2 + x_2^2\right)$$
$$x_1^3 - x_2^3 = (x_1 - x_2)\left((x_1 - x_2)^2 + 3x_1x_2\right).$$

Additionally, we will use Vieta's Theorem for the product of the roots as

$$x_1 \cdot x_2 = q$$
$$x_1 + x_2 = -p$$

Substituting the second set of equalities into the first formula and using the condition of the problem, such that $x_1 - x_2 = 5$ and $x_1^3 - x_2^3 = 35$, we obtain

$$35 = 5 \cdot \left(5^2 + 3 \cdot q\right)$$
$$3q = 7 - 25$$
$$q = -6.$$

On the other hand, from the difference of squares the following is also true:

$$(x_1 + x_2)^2 - (x_1 - x_2)^2 = 4x_1 \cdot x_2$$

This relationship gives us the connection between parameters p and q of the quadratic equation

$$(-p)^2 - (5)^2 = 4q$$
$$p^2 = 4q + 25 = 4 \cdot (-6) + 25 = 1$$
$$p = 1 \text{ or } p = -1$$

Therefore we have two possible pairs for parameters p and q:

$$(p, q) = \{(1, -6); (-1, -6)\}$$

It is interesting that using Vieta's Theorem, the difference of squares, and difference of cubes formulas, we were able to find the answer without finding the roots (2, −3) and (3, −2) of the equations $x^2 + x - 6 = 0$ and $x^2 - x - 6 = 0$, respectively.

Answer $(p, q) = \{(1, -6); (-1, -6)\}$.

Problem 51 Solve the equation $x^2 + x = 1111111122222222$.

Solution We are not going to solve it using a quadratic formula because the number on the right side is too big. Instead, we will factor the left side and rewrite the big number in a standard form with base 10:

$$\begin{aligned} x(x+1) = 1 \cdot 10^{16} + 1 \cdot 10^{15} + \ldots + 1 \cdot 10^{10} + 1 \cdot 10^{9} \\ + 2\left(10^{8} + 10^{7} + \ldots + 10^{2} + 10 + 1\right) \end{aligned}$$

Next, we can regroup terms on the right-hand side and obtain

$$\begin{aligned} x(x+1) &= \left(1 \cdot 10^{16} + 1 \cdot 10^{15} + \ldots + 1 \cdot 10^{2} + 1 \cdot 10 + 1\right) \\ &\quad + \left(10^{8} + 10^{7} + \ldots + 10^{2} + 10 + 1\right) \\ &= \frac{10^{16} - 1}{10 - 1} + \frac{10^{8} - 1}{10 - 1} \end{aligned}$$

Here we applied the formula for the sum of a geometric series twice, with common ratio 10, first term 1 with 16 and 8 terms, respectively.

Next, we can further simplify the right-hand side by applying the difference of squares to the first term and factoring the common factor:

$$\begin{aligned} x(x+1) &= \frac{10^{16} - 1}{9} + \frac{10^{8} - 1}{9} \\ &= \frac{\left(10^{8} - 1\right)\left(10^{8} + 1\right)}{3 \cdot 3} + \frac{10^{8} - 1}{3} \cdot \frac{1}{3} \\ &= \frac{10^{8} - 1}{3}\left(\frac{10^{8} + 1 + 1}{3}\right) = \frac{10^{8} - 1}{3}\left(\frac{10^{8} - 1 + 3}{3}\right) \\ &= \frac{10^{8} - 1}{3}\left(\frac{10^{8} - 1}{3} + 1\right) \end{aligned}$$

It follows from the last equality that one of the roots of the given quadratic equation is

$$x_1 = \frac{10^{8} - 1}{3} = 33333333.$$

Then by Vieta's Theorem, the second root is

$$x_2 = -\frac{10^{8} - 1}{3} - 1 = -33333334.$$

Answer The roots are 33333333 and -33333334.

2.2.2 *Interesting Facts About Quadratic Functions and Their Roots*

Many challenging problems in elementary mathematics are related to location of the zeroes of a quadratic function on the X-axis but can be solved without actual evaluation of these zeroes. Below are some examples that can be very helpful.

Case 1 Figure 2.2 demonstrates the case when **both roots of quadratic function are greater than a given number** (number M for the red curve and number N for the blue). For example, this condition for the red parabola and point M can be written as

$$\begin{cases} D > 0 \\ x_v > M \\ a \cdot f(M) > 0 \end{cases}$$

The same set of the inequalities can be obtained for the blue parabola and point N.

Case 2 **Both roots of a quadratic function lie inside segment** [$\boldsymbol{M}$, $\boldsymbol{N}$] (see Figure 2.3).

The following system is valid:

$$\begin{cases} D > 0 \\ x_v \in [M, N] \\ a \cdot f(M) \geq 0 \\ a \cdot f(N) \geq 0 \end{cases}$$

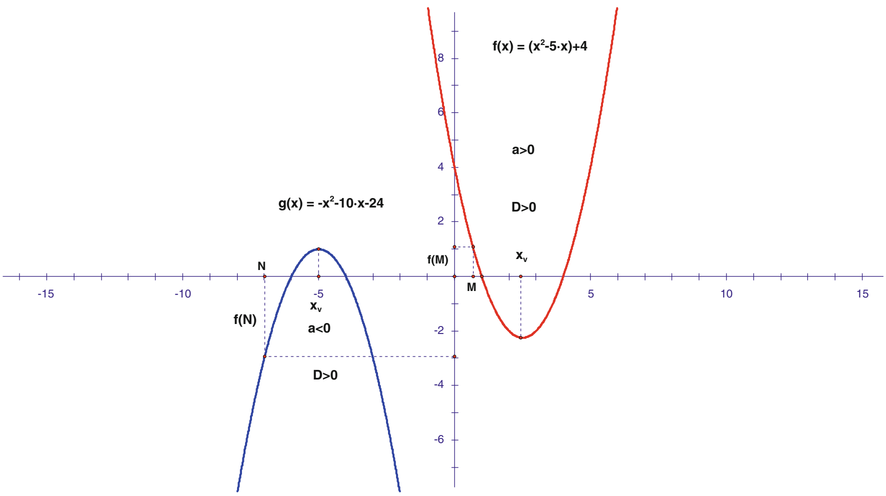


Figure 2.2 Case 1 for quadratic functions

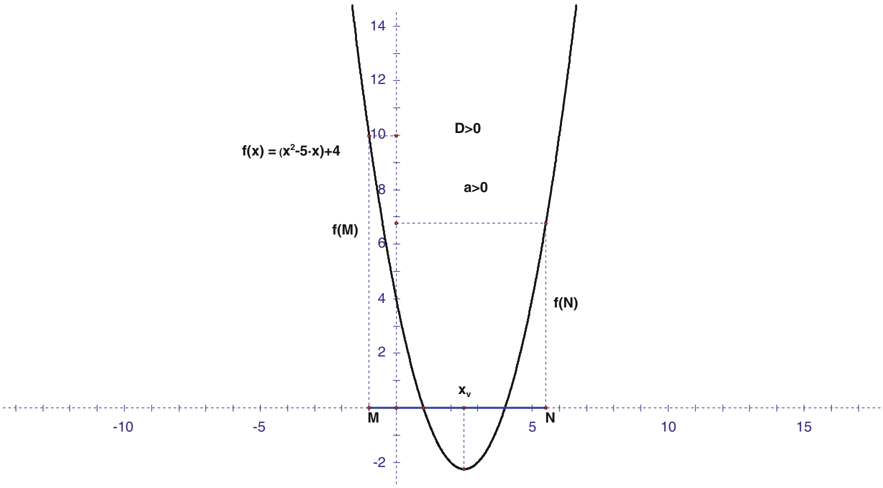


Figure 2.3 Case 2 for quadratic functions

Below we show a parabola with positive leading coefficient. However, the same system will be valid for a negative leading coefficient. Sketch it yourself and you will see that both products, $a \cdot f(M)$ and $a \cdot f(N)$, will be greater than or equal to zero as products of two negative numbers.

Case 3 **One of the zeroes of a quadratic function is inside of the given interval [M, N] and the other zero is to the left of the given interval**:

$$\begin{cases} D > 0 \\ x_v \in [M, N] \\ a \cdot f(M) < 0 \\ a \cdot f(N) > 0 \end{cases}$$

For illustration we can consider parabola with a negative leading coefficient (Figure 2.4). You can show that the set of the relationships is valid for any sign of the leading coefficient.

Case 4 **Quadratic function has two real zeroes and one of them is inside of the given interval [*M*, *N*] and the other is to the right of the interval**.

The following is true:

$$\begin{cases} D > 0 \\ x_v \in [M, N] \\ a \cdot f(M) > 0 \\ a \cdot f(N) < 0 \end{cases}$$

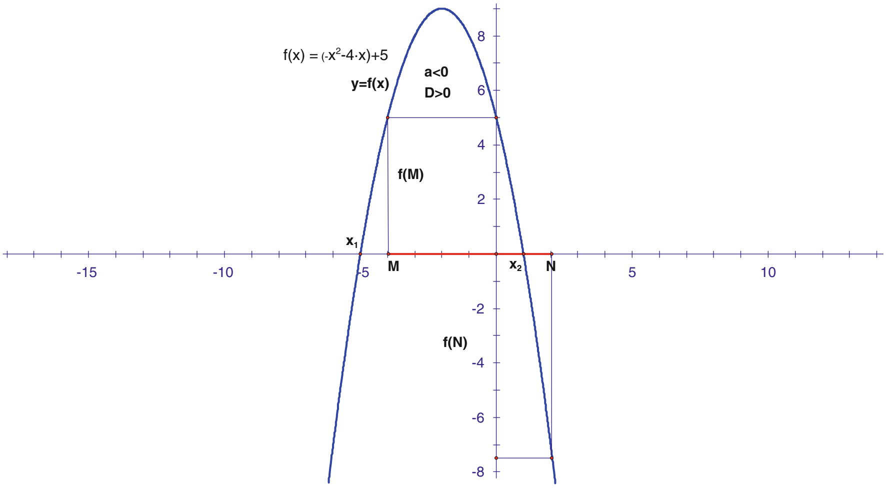


Figure 2.4 Case 3 for quadratic functions

Please sketch the parabola yourself and make sure that you understand it, and that now you can consider all other cases that we omitted here and left for you as an exercise.

Problem 52 (MGU Exam 1996). Find the sum of all integer values of a parameter a for which function $f(x) = x^2 + 2ax + a^2 + 4a$ is negative for all $x \in (1, 3)$.

Solution This parabola opens upward, and it can have negative behavior for all x from the given interval if and only if it has two zeroes x_1, x_2 such that if $x_2 > x_1$, then $x_2 > 3, x_1 < 1$. Moreover, $\frac{D}{4} > 0, f(1) < 0, f(3) < 0$. We have the system

$$\begin{cases} \dfrac{D}{4} = a^2 - \left(a^2 + 4a\right) = -4a > 0 \\ f(1) = a^2 + 6a + 1 < 0 \\ f(3) = a^2 + 10a + 9 < 0 \end{cases}$$

Using D/4 formula for both trinomials, we obtain

$$\begin{cases} a < 0 \\ -3 - 2\sqrt{2} \le a \le -3 + 2\sqrt{2} \\ -9 \le a \le -1 \end{cases} \Leftrightarrow \quad a \in \left[-3 - 2\sqrt{2}, \; -1\right]$$

The following natural numbers belong to this interval: $a = \{-5, -4, -3, -2, -1\}$. Adding all these numbers gives us the answer.

Answer -15.

Problem 53 Find all values of a parameter a at which the minimal value of the function $y = x(x - 1 - a) - a^2 + 3a + 7$ on the interval [0, 2] equals 2.

Solution This function can be written as

$$y = x^2 - (a + 1)x - a^2 + 3a + 7$$

The graph of this function is a parabola that is opened upward and it will have an absolute minimum at its vertex,

$$x_v = \frac{a + 1}{2}. \tag{2.4}$$

However, we need to look for the minimal value on the given interval.
The following cases are possible:

1. Vertex of the parabola $x_v \in [0, 2]$ and the value of the function at x_v equals 2:

$$x^2 - (a + 1)x - a^2 + 3a + 7 = 2$$

or

$$x^2 - (a + 1)x - a^2 + 3a + 5 = 0 \tag{2.5}$$

Because this equation must have only one root (the value of 2 must occur at the vertex of a parabola) then the discriminant of this quadratic equation must be zero!

$$D = (a + 1)^2 + 4\left(a^2 - 3a - 5\right) = 0$$

or

$$5a^2 - 10a - 19 = 0$$

This occurs at two values of parameter a:

$$a_1 = \frac{5 - 2\sqrt{30}}{5}, \quad a_2 = \frac{5 + 2\sqrt{30}}{5}$$

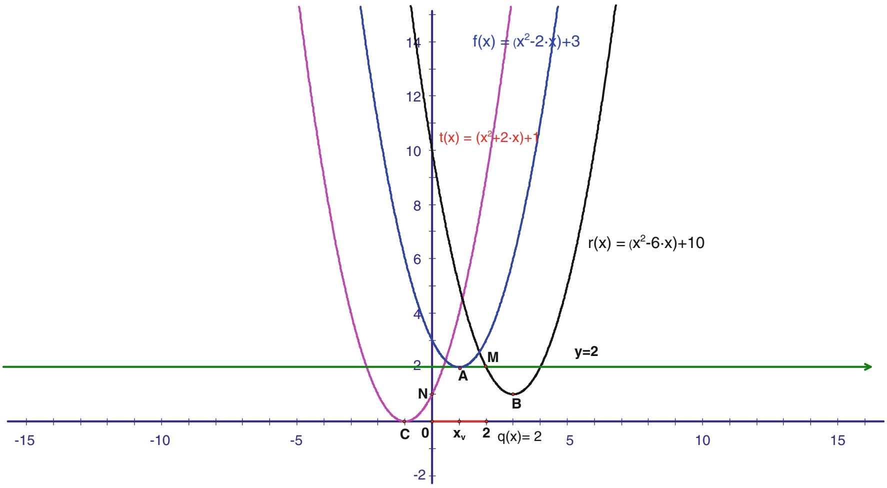


Figure 2.5 Sketch for Problem 53

However, if we substitute either of these values into the formula (2.4) for the vertex, we obtain that $x_v(a_1) < 0,\ x_v(a_2) > 2$.

This contradicts our assumption that the parabola's vertex belongs to the given interval (blue curve in Figure 2.5).

2. Consider now the case illustrated by the black curve. The coordinate of its vertex (B) is greater than 2 and the function is decreasing on the given interval [0, 2]; then it approaches its minimal value on [0, 2] at $x = 2$. (See point M on the graph.)

   Replacing x by 2 into (2.5) we obtain the following quadratic equation:

$$a^2 - a - 7 = 0$$
$$a_1 = \frac{1 - \sqrt{29}}{2}, \quad a_2 = \frac{1 + \sqrt{29}}{2}$$

   If we substitute each of the a values into (2.4), we obtain that only at the second value the vertex of the parabola will lie to the right of $x = 2$:

$$x_v(a_2) = \frac{1 + \sqrt{29} + 2}{4} = \frac{3 + \sqrt{29}}{4} > 2$$

   Therefore, $a = \dfrac{1 + \sqrt{29}}{2}$ is one of the solutions.

3. Consider the third and the last possible cases illustrated by the purple curve. This parabola is increasing on the interval [0, 2] and has its vertex (C) located to the left of 0. Therefore, its minimal value on [0, 2] will occur at $x = 0$ (point N in Figure 2.5).

If we substitute $x = 0$ into (2.5) we obtain

$$a^2 - 3a - 5 = 0$$
$$a_1 = \frac{3 - \sqrt{29}}{2}, \quad a_2 = \frac{3 + \sqrt{29}}{2}$$

Substituting both values of $\boldsymbol{a}$ into vertex formula (2.4) we obtain that only at $a = \frac{3 - \sqrt{29}}{2}$, $x_v = \frac{3 - \sqrt{29} + 2}{4} = \frac{5 - \sqrt{29}}{4} < 0$ (the vertex is located to the left of the left boundary of the interval, 0). Therefore this value for the parameter a will also be an answer.

Answer $a = \frac{1 + \sqrt{29}}{2}; \frac{3 - \sqrt{29}}{2}$.

Problem 54 For what values of a parameter a does one of the roots of quadratic function $f(x) = 4x^2 - 4x - 3a$ lie in the interval $[-1, 1]$?

Solution Note that if one of the roots belongs to the given interval, $[a, b]$, then the quadratic function takes opposite values at the ends of the given interval. This can be written as

$$f(-1) \cdot f(1) \le 0$$
$$f(-1) = 4 \cdot (-1)^2 - 4 \cdot (-1) - 3a = 8 - 3a$$
$$f(1) = 4 \cdot (1)^2 - 4 \cdot (1) - 3a = -3a$$
$$(3a - 8) \cdot 3a \le 0$$
$$0 \le a \le \frac{8}{3}$$

Answer $0 \le a \le \frac{8}{3}$.

Remark Please notice that while solving the above problem, we did not use the positiveness of discriminant. Explain why that was not necessary in order to get the answer.

Problem 55 Find all solutions of the given equation
$\left|x^2 - 2|x| + 1\right| = 3|2 - x| - 1$.

Solution Let us rewrite the equation in a different form by completing the square on the left:

$$(|x| - 1)^2 = 3 \cdot |x - 2| - 1 \tag{2.6}$$

And consider this equation as

$$f(x) = g(x)$$

on each on the following intervals $\left(-\infty, 0\right] \cup \left(0, 2\right] \cup (2, \infty)$:

1. If $x \in \left(-\infty, 0\right]$ then $|x| = -x$ and $|x - 2| = 2 - x$
   Then (2.6) takes the form

   $$x^2 + 5x - 4 = 0,$$

   which has to be solved under the condition $x \leq 0$. Hence out of the two roots

   $$x_1 = \frac{-5 - \sqrt{41}}{2}, \quad x_2 = \frac{-5 + \sqrt{41}}{2}$$

   only the first one, $\frac{-5 - \sqrt{41}}{2}$, works.
2. If $0 < x \leq 2$, then (2.6) becomes

   $$x^2 + x - 4 = 0$$
   $$x_1 = \frac{-1 - \sqrt{17}}{2}, \quad x_2 = \frac{-1 + \sqrt{17}}{2}$$

   Only $x = \frac{-1 + \sqrt{17}}{2} \in \left(0, 2\right]$.
3. If $x > 2$, then (2.6) has the form

   $$x^2 - 5x + 8 = 0$$

   Because the discriminant of this equation is less than zero, this equation does not have real roots.

Answer The equation has two solutions: $x = \frac{-5 - \sqrt{41}}{2}$ and $x = \frac{-1 + \sqrt{17}}{2}$.

Next, I want to offer you a problem that will use algebra and geometry.

Problem 56 Consider all quadratic functions $f(x) = x^2 + px + q$ that intersect the coordinate system at three points. Prove that all circles that pass through each of such three points have a common point. Find its coordinates.

Proof If a parabola with positive leading coefficient intersects the coordinate system at three points, then it has a positive discriminant. Assume that two of these points are X-intercepts $A(x_1, 0), B(x_2, 0)$ and the third one is the Y-intercept, $C(0, q)$. For different values of the parameters in the quadratic function the origin is either between the two intercepts or on one side of both X-intercepts. Consider both cases shown in Figure 2.6. We need to prove that point G is that common point of all possible circles.

Case 1 Let us connect points A and C, C and B, A and B, A and G, and G and B (see Figure 2.7).

Because the angles ACO and GBA are the same (they subtend the same arc AG), triangles AOC and GOB are similar. Therefore, $\frac{|AO|}{|OG|} = \frac{|OC|}{|OB|}$. Denote $|OG| = y$, then

$$\frac{|x_1|}{|y|} = \frac{|q|}{|x_2|}$$

Solving which for y will give us

$$x_1 x_2 = q \cdot y$$

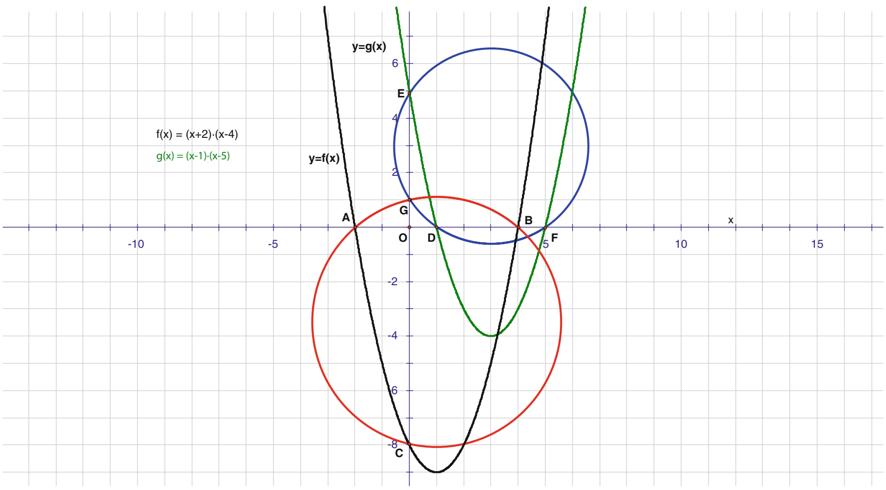


Figure 2.6 Sketch for Problem 56

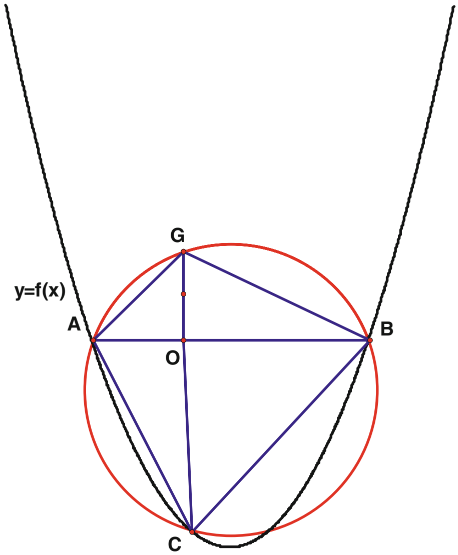


Figure 2.7 Case 1 (Problem 56)

On the other hand, from Vieta's Theorem we have that

$$x_1 x_2 = q$$

It follows from these two equations that $y = 1$, and that $G(0,1)$.

Case 2 Let us draw the second parabola and form a red cyclic quadrilateral by connecting G, E, F, and D (see Figure 2.8). We need to prove that in this case point G is the same as in Case 1, i.e., $G(0,1)$.

In order to prove this, we will erase our parabola and the coordinate system as it is shown in Figure 2.9. Angles GDO and GDF are supplementary angles. Quadrilateral $EFDG$ is cyclic, so the angles GDF and GEF are also supplementary. Therefore, $\angle GDO = \angle GEF$ (shown by green arc).

Because the angle EOF is the right angle triangles GDO and EOF are similar right triangles, where $|OF| = |x_2|, |OD| = x_1,\ |OE| = q$ and $OG = y$. The following is valid:

$$\frac{|x_1|}{|y|} = \frac{|q|}{|x_2|} \Leftrightarrow y = \frac{x_1 x_2}{q} = \frac{q}{q} = 1 = |OG|$$

The proof is complete and all circles pass through the point $G(0,1)$.

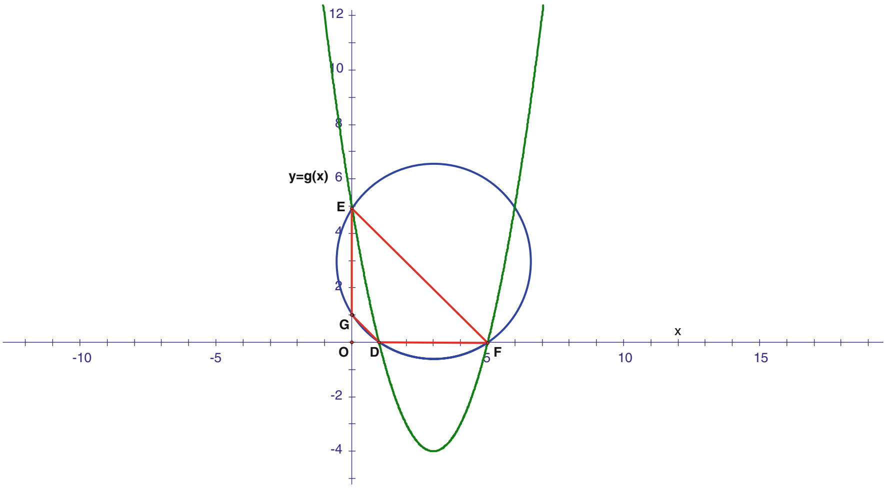
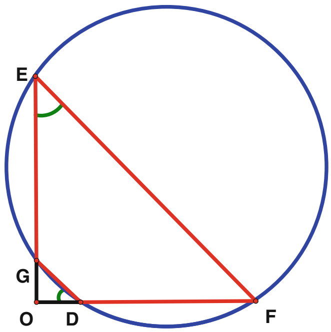


Figure 2.8 Case 2 (Problem 56)

Figure 2.9 Similar triangles (Problem 56)



2.3 Polynomial Equations in Two or Three Variables

Equations in integers or natural numbers are often used at different mathematics contests. Very often students are afraid to even start solving such problems. Some techniques will be taught here. In order to introduce you to the topic, I want to offer the following problems.

Problem 57 Can a quadratic equation $ax^2 + bx + c = 0$ with integer coefficients have discriminant 23?

Solution Assume that discriminant is 23, i.e., $b^2 - 4ac = 23$.

Adding 2 to both sides and moving $4ac$ to the right-hand side, we rewrite it as

$$b^2 - 25 = 4ac - 2$$

Applying the difference of squares to the left side and factoring the right side, we have

$$(b-5)(b+5) = 2(2ac-1) \tag{2.7}$$

The two factors on the left-hand side differ by 10 and they are either both odd or both even. If the factors are odd, then the equation has no solution.

If the factors are even, then the left side is divisible by 4. The right side, in turn, is even, but not divisible by 4. Equation (2.7) has no solution in integers. Since (2.7) has no solution, its discriminant cannot be 23.

Answer No.

Problem 58 (USSR Olympiad 1986) It is known that the roots of the equation $x^2 + ax + 1 = b$ are natural numbers. Prove that $a^2 + b^2$ is not a prime number.

Proof 1 (Using Vieta's Theorem)

Let us rewrite the equation as

$$x^2 + ax + 1 - b = 0$$

By Vieta's Theorem we have

$$x_1 \cdot x_2 = 1 - b$$
$$x_1 + x_2 = -a$$

Solving for a and b

$$\begin{aligned}
&b = 1 - x_1 \cdot x_2 \\
&a = -(x_1 + x_2) \\
&b^2 = (1 - x_1 \cdot x_2)^2 \\
&a^2 = (x_1 + x_2)^2 \\
&a^2 + b^2 = {x_1}^2 + {x_2}^2 + 1 + (x_1 x_2)^2 \\
&\qquad\quad = {x_2}^2\left(1 + x_1^2\right) + \left(1 + x_1^2\right) \\
&\qquad\quad = \left(1 + x_1^2\right)\left(1 + x_2^2\right)
\end{aligned}$$

Because the sum of the squares is a product of two quantities, it is a composite number. Therefore, it is not prime.

Proof 2 (Using complex conjugate polynomials)
Consider $p(x) = x^2 + ax + (1 - b)$, then

$$\begin{aligned} p(i) \cdot p(-i) &= \left(i^2 + a \cdot i + 1 - b\right) \cdot \left((-i)^2 - ai + 1 - b\right) \\ &= (ai - b) \cdot (-ai - b) \\ &= a^2 + b^2 \end{aligned} \tag{2.8}$$

On the other hand, if x_1, x_2 are natural roots of the quadratic equation $x^2 + ax + 1 - b = 0$, then the quadratic function can be written as

$$p(x) = (x - x_1)(x - x_2)$$

Substituting $x = i, x = -i$, we obtain the following:

$$\left.\begin{aligned} p(i) &= (i - x_1) \cdot (i - x_2) \\ p(-i) &= (-i - x_1) \cdot (-i - x_2) \end{aligned}\right\} \Rightarrow$$

$$p(i)\,p(-i) = (-x_1 + i)(-x_1 - i)(-x_2 + i)(-x_2 - i)$$

$$p(i) \cdot p(-i) = \left(x_1^2 + 1\right)\left(x_2^2 + 1\right) \tag{2.9}$$

Equating the right sides of (2.8) and (2.9), we obtain that

$$a^2 + b^2 = \left(1 + x_1^2\right) \cdot \left(1 + x_2^2\right)$$

Because the sum of the squares is the product of two numbers, each different from one, the sum of the squares cannot be a prime number.

2.3.1 *Special Products*

Many equations in two or three variables that are subject to solution over the set of all integers can be essentially simplified if we apply factoring. In order to refresh your factoring skills I want to give you the most important formulas called special products.

A Difference of Squares

A difference of squares can be factored as

$$u^2 - v^2 = (u - v)(u + v) \tag{2.10}$$

whatever we have for u and v.

Example $(x-1)^2 - 9 = (x-1)^2 - 3^2 = (x-1-3)(x-1+3) = (x-4)(x+2)$
Here $u = x - 1$ and $v = 3$.

Many students now are very addicted to a calculator. If they do not have a calculator, they panic and start using their cell phone calculator. Most of them still know their multiplication tables and squares of simple numbers. So I tell the following story: Suppose you like windsurfing and, seeking the best wave, you come to a Pacific island X where some aborigines are still practicing cannibalism. You appear to be in a small village of these "cruel" people who are preparing to eat you. The only thing that can save your life is to give them the answer right away:

What is 3999*4001?

Students who do not know the difference of squares formula would not get out alive . . ..

Those who notice that

$$3999 \cdot 4001 = (4000 - 1)(4000 + 1)$$

would get the answer immediately as

$$(4000)^2 - 1 = 16000000 - 1 = 15999999.$$

Of course, the story is a joke. However, formulas above and those below are created and proven for us in order to make our mathematical experience a more pleasant journey. By the way, this formula for the difference of squares was known to ancient Greeks, who used geometric approach for its proof.

Proof Consider a big square *ABCD* with side *a* and a small red square *EBGF* with side *b* (see Figure 2.10).

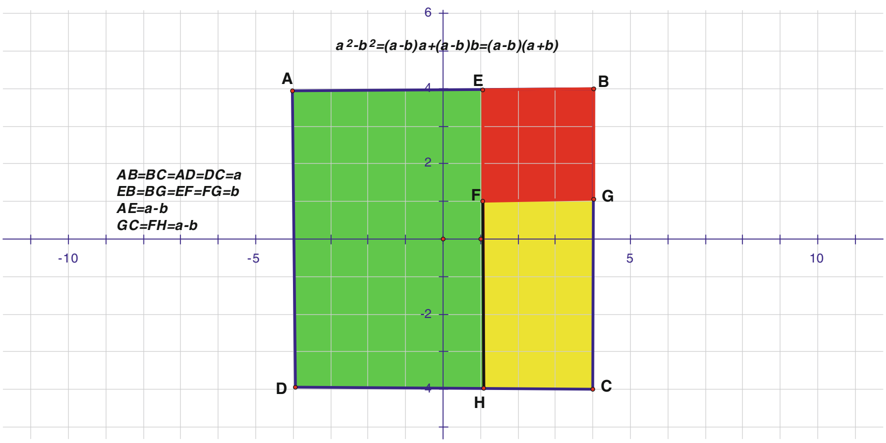


Figure 2.10 Big and small squares

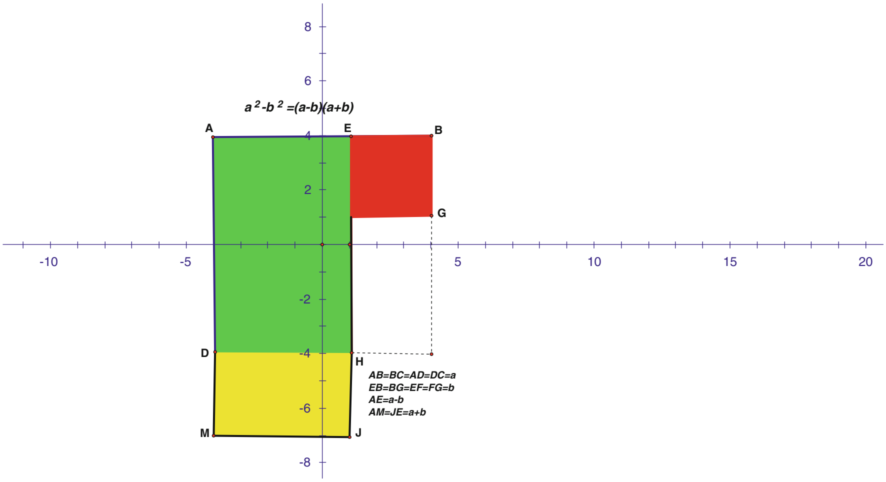


Figure 2.11 Difference of squares

If we cut out the red square, we will obtain a geometric figure containing a green rectangle with sides $(a-b)$ and a, and the yellow rectangle with sides b and again $(a-b)$. In order to find the sum of the areas of both rectangles we can rearrange them in such a way that they will touch each other by the side of the same length $(a-b)$. Thus, a new rectangle will be formed with one side $(a-b)$ and the other $(a+b)$, the area of which is $(a-b)(a+b)$ (see Figure 2.11).

By taking the area of b^2 away from the larger area of a^2, or in other words, finding $a^2 - b^2$, we have found that the area is also equal to $(a-b)(a+b)$. Done.

Difference of Cubes

$$u^3 - v^3 = (u - v)\left(u^2 + uv + v^2\right) \tag{2.11}$$

Sum of Cubes

$$u^3 + v^3 = (u + v)\left(u^2 - uv + v^2\right) \tag{2.12}$$

Difference of *n*th Powers

$$u^n - v^n = (u - v)\left(u^{n-1} + u^{n-2}v + u^{n-3}v^2 + \ldots + u^2v^{n-3} + uv^{n-2} + v^{n-1}\right) \tag{2.13}$$

If $n = 2k$ (an even power), then

$$
\begin{aligned}
u^n - v^n = u^{2k} - v^{2k} &= \left(u^k - v^k\right)\left(u^k + v^k\right) \\
&= \left(u^2\right)^k - \left(v^2\right)^k = \left(u^2 - v^2\right) \cdot \ p_{(2k-2)} \\
&= \ (u - v)(u + v) \cdot p_{(2k-2)}
\end{aligned}
\tag{2.14}
$$

where $p_{(2k-2)}$ is a polynomial of $(2k - 2)$ degree.

This formula plays a very important role in solving problems on integers and divisibility.

A Trinomial Square

$$
\begin{aligned}
u^2 + 2uv + v^2 &= (u + v)^2 \\
u^2 - 2uv + v^2 &= (u - v)^2
\end{aligned}
\tag{2.15}
$$

Example $x^2 + 1 - 2x = x^2 - 2x \cdot 1 + 1^2 = (x - 1)^2$.

Factoring Quadratic Function

Any quadratic equation with zeros x_1 and x_2 can be factored as

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

Example Factor $x^4 - 2x^2 - 8$.

Let $u = x^2$; then the given function becomes quadratic with respect to u:

$$u^2 - 2u - 8 = (u - 4)(u + 2) = \left(x^2 - 4\right)\left(x^2 + 2\right) = (x - 2)(x + 2)\left(x^2 + 2\right)$$

To complete factoring we applied the formula of the difference of squares as well.

Some problems involving numbers contain exponential expressions, so we have to list two of the most useful properties of exponents. We utilized these formulas earlier in this chapter.

Cube of a Binomial

$$
\begin{aligned}
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a + b) \\
(a - b)^3 &= a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a - b)
\end{aligned}
\tag{2.16}
$$

Problem 59 Prove that $\sqrt[3]{6+\sqrt{\frac{847}{27}}}+\sqrt[3]{6-\sqrt{\frac{847}{27}}}$ is a rational number.

Proof Assume that $x=\sqrt[3]{6+\sqrt{\frac{847}{27}}}+\sqrt[3]{6-\sqrt{\frac{847}{27}}}$ and let us cube both sides using the formula above:

$$x^3=12+3x\cdot\sqrt[3]{6^2-\frac{847}{27}}.$$

This can be written as $x^3-5x-12=0$.

By the Rational Zero Theorem, we can find that $x=3$ is a zero of the equation. Continuing factoring it, we get

$$x^3-5x-12=(x-3)\left(x^2+3x+4\right)=0$$

The second factor does not have any real zeroes.

Therefore, $x=\sqrt[3]{6+\sqrt{\frac{847}{27}}}+\sqrt[3]{6-\sqrt{\frac{847}{27}}}=3$ is a rational number.

Useful Formula

$$a^3+b^3+c^3-3abc=(a+b+c)\left(a^2+b^2+c^2-ab-bc-ac\right) \tag{2.17}$$

Factoring of an Exponent

$$\begin{aligned} a^{n+m}&=a^n\cdot a^m\\ (a^n)^m&=a^{n\cdot m}\end{aligned} \tag{2.18}$$

Example $4^{2x}-1=(4^x)^2-1^2=(4^x-1)(4^x+1)$.

Doing this problem, we used properties of exponents and again a difference of squares.

Factoring the Sum of Squares

$$(ac+bd)^2+(ad-bc)^2=\left(a^2+b^2\right)\left(c^2+d^2\right) \tag{2.19}$$

Please prove this formula yourself. The usage of this formula is very important when you have to rewrite a number as sum of squares of other numbers. Though this formula is true for any a, b, c, and d, not every integer can be written as a sum of two squares.

Homogeneous Polynomials

If a polynomial $f(x,y)$ possesses the property $f(tx,ty) = t^n f(x,y)$ for some natural number n, then $f(x,y)$ is said to be a homogeneous polynomial of degree n in two variables x, y.

For example, $f(x,y) = x^3 + 2x^2y + y^3$ is a homogeneous polynomial of degree 3 because

$$f(tx,ty) = (tx)^3 + 2(tx)^2(ty) + (ty)^3 = t^3\left(x^3 + 2x^2y + y^3\right) = t^3 f(x,y)$$

There exists a general approach of factoring homogeneous polynomials as

$$f(x,y) = y^n f(1,u), \text{ where } u = \frac{x}{y} \tag{2.20}$$

Let us consider a homogeneous polynomial of the 4th degree:

$$f(x,y) = 6x^4 + 25x^3y + 12x^2y^2 - 25xy^3 + 6y^4 \tag{2.21}$$

Next, we apply (2.20) to it. For this we factor out y^4:

$$\begin{aligned} f(x,y) &= y^4\left[6\left(\frac{x}{y}\right)^4 + 25\left(\frac{x}{y}\right)^3 + 12\left(\frac{x}{y}\right)^2 - 25\frac{x}{y} + 6\right] \\ &= y^4[6u^4 + 25u^3 + 12u^2 - 25u + 6] \end{aligned} \tag{2.22}$$

If polynomial (2.21) can be factored, then the polynomial within brackets must have rational zeroes. Using the Rational Zero Theorem we find $u = -2$, $u = -3$, $u = 1/2$, and $u = 1/3$. Now the polynomial (2.22) can be factored as

$$\begin{aligned} f(x,y) &= y^4[(u+3)(u+2)(3u-1)(2u-1)] \\ &= [y(u+3)][y(u+2)][y(3u-1)][y(2u-1)] \end{aligned}$$

Replacing $u = x/y$ into above, we obtain that $f(x,y) = (x+3y)(x+2y)(3x-y)(2x-y)$.

Remark I have to mention here that some homogeneous polynomials of second degree can be factored mentally using FOIL as

$$ax^2 + bxy + cy^2 = (mx + py)(nx + sy),$$

if we can find m, n, p, and s such that $mn = a$, $ps = c$, and $ms + pn = b$

Thus, $15x^2 - 11xy + 2y^2 = (5x - 2y)(3x - y)$, because
$5 \cdot 3 = 15, \;\; (-2)(-1) = 2$ and $5 \cdot (-1) + (-2) \cdot 3 = -11.$

Special Property of Homogeneous Equations of Even Degree

For a homogeneous equation of second, fourth, sixth, and so on (any even degree) if (a, b) is a solution, then $(-a, -b)$ will be a solution as well. From the definition of a homogeneous polynomial we find that $f(-a, -b) = (-1)^n \cdot f(a, b)$. But if $n = 2k$ (even degree), then $f(-a, -b) = f(a, b)$.

Therefore, if (a, b) is a solution, then $(-a, -b)$ is another solution.

Summary Every time you have a problem in integers, first try to factor it. For this, you have to be able to recognize a special product or rewrite the given expression such that you can use a special product.

Problem 60 (Sophie Germain) Prove that $n^4 + 4$ cannot be prime, if $n > 1$.

Proof Let us show that this number can be factored. Thus,

$$\begin{aligned} n^4 + 4 &= n^4 + 4 - 4n^2 + 4n^2 = n^4 + 4n^2 + 4 - (2n)^2 = (n^2 + 2)^2 - (2n)^2 \\ &= (n^2 + 2n + 2)(n^2 - 2n + 2) = \left((n+1)^2 + 1\right)\left((n-1)^2 + 1\right) \end{aligned}$$

Therefore this number is a product of two other numbers and of course, it cannot be prime.

Problem 61 DeBouvelles (1509) claimed that one or both of $(6n + 1)$ and $(6n - 1)$ are primes for all positive integers. Show that there are infinitely many n such that $(6n - 1)$ and $(6n + 1)$ are composite.

Solution Because the sum and the difference of cubes can be factored, let $n = 36m^3$.

$$\begin{aligned} 6n + 1 = 6 \cdot 36m^3 + 1 = (6m)^3 + 1^3 = (6m + 1)(36m^2 - 6m + 1) \\ 6n - 1 = 6 \cdot 36m^3 - 1 = (6m)^3 - 1^3 = (6m - 1)(36m^2 + 6m + 1) \end{aligned}$$

For example,

$$\begin{aligned} &m = 1, n = 36, 6n - 1 = 215 = 5 \cdot 43; \;\; 6n + 1 = 217 = 7 \cdot 31 \\ &m = 2, n = 288, 6n - 1 = 1727 = 11 \cdot 15 \cdot 7; \;\; 6n + 1 = 1729 = 7 \cdot 13 \cdot 19 \end{aligned}$$

There are other possible values for n that allow us to factor $(6n+1)$ or $(6n-1)$.

For example, if $n = 6k^2,\ k \in N \ \Rightarrow\ 6n - 1 = 36k^2 - 1 = (6k-1)\cdot(6k+1)$.

Some such numbers are shown below. Note that the above substitution will not work for $(6n+1)$:

$$
\begin{aligned}
&k = 1,\ n = 6,\ \ 6n - 1 = 35 = 5 \cdot 7 \\
&k = 2,\ n = 24,\ 6n - 1 = 143 = 11 \cdot 13 \\
&k = 3,\ n = 54,\ 6n - 1 = 323 = 17 \cdot 19
\end{aligned}
$$

Problem 62 (American Mathematical Monthly 1977) The difference of two consecutive cubes is a perfect square of some number. Prove that this number can be represented as the sum of two consecutive squares.

Proof Let us consider the difference of two consecutive cubes:

$$(x+1)^3 - x^3 = 3x^2 + 3x + 1 = y^2$$

Here y is unknown and we must prove that it can be written as the sum of two consecutive squares. Multiply both sides by 4:

$$
\begin{aligned}
4(3x^2 + 3x + 1) &= 4y^2 \\
12x^2 + 12x + 4 &= 4y^2
\end{aligned}
$$

Completing the square on the left and moving one to the right-hand side:

$$
\begin{aligned}
&3(4x^2 + 4x + 1) + 1 = 4y^2 \\
&3(2x+1)^2 = 4y^2 - 1 \\
&3(2x+1)^2 = (2y-1)(2y+1)
\end{aligned}
$$

Because $(2y-1, 2y+1)$ are relatively prime, we have two possible cases:
Case 1

$$
\begin{aligned}
&2y - 1 = 3m^2,\ (n, m) = 1 \\
&2y + 1 = n^2,\quad 4y^2 = n^2 + 3m^2,
\end{aligned}
$$

Case 2

$$
\begin{aligned}
&2y - 1 = m^2,\quad (n, m) = 1 \\
&2y + 1 = 3n^2,\quad 4y^2 = 3n^2 + m^2,
\end{aligned}
$$

Of course, m and n are odd integers.

Case 1 is not possible because it leads to the equation

$$n^2 - 3m^2 = 2$$

which does not have a solution in integers (the square of any number divided by 3 leaves a remainder of 0 or 1, not 2).
From case 2 we have

$$\begin{aligned}
&2y - 1 = m^2, \quad (n, m) = 1\\
&2y = m^2 + 1\\
&2y = (2k + 1)^2 + 1\\
&2y = 4k^2 + 4k + 2 = 2\left(k^2 + 2k + 1 + k^2\right)\\
&2y = 2\left[(k + 1)^2 + k^2\right]\\
&y = (k + 1)^2 + k^2
\end{aligned}$$

The last formula represents y as the sum of two consecutive squares.

Problem 63 Find all integers x and y that satisfy the equation $xy = x + y$.

Solution Let us rewrite the equation in the form

$$xy - x - y + 1 = 1$$

Factoring by grouping the left-hand side we have

$$\begin{aligned}
&x(y - 1) - (y - 1) = 1\\
&(y - 1)(x - 1) = 1
\end{aligned}$$

We have obtained a very interesting situation. Because a number on the right side is one, we have only two opportunities.

If $y - 1 = 1$ and $x - 1 = 1$, then $y = 2$ and $x = 2$.
If $y - 1 = -1$ and $x - 1 = -1$, then $x = 0$ and $y = 0$.

Answer (0, 0) and (2, 2).

Problem 64 Find all primes x and y (positive and negative) that satisfy the equation $2x^3 + xy - 7 = 0$.

Solution This problem may look unusual, because it has one equation and two variables x and y. Of course, you noticed that 7 is a prime number. Using ideas from the previous problem, we can see the advantage of moving 7 to the right-hand side of the equation

$$2x^3 + xy = 7$$

The left side of this equation can be factored as

$$x\left(2x^2 + y\right)$$

The right side (number 7) in turn can be factored as follows:

$$\begin{aligned} 7 &= 1 \times 7 \\ 7 &= 7 \times 1 \\ 7 &= (-1) \times (-7) \\ 7 &= (-7) \times (-1) \end{aligned}$$

Next, we have four possible cases for numbers x and y that are reducible to four systems:

$$\text{Case 1} \begin{cases} x = 1 \\ 2x^2 + y = 7 \end{cases} \qquad x = 1,\ y = 5.$$

$$\text{Case 2} \begin{cases} x = 7 \\ 2x^2 + y = 1 \end{cases} \qquad x = 7,\ y = -97.$$

$$\text{Case 3} \begin{cases} x = -1 \\ 2x^2 + y = -7 \end{cases} \qquad x = -1,\ y = -9.$$

$$\text{Case 4} \begin{cases} x = -7 \\ 2x^2 + y = -1 \end{cases} \qquad x = -7,\ y = -99.$$

We find four ordered pairs of (x,y) that satisfy the given equation, but among them only the pair of primes $(7, -97)$ $x = 7$ and $y = -97$ satisfies the condition of the problem.

Answer $x = 7$ and $y = -97$.

Problem 65 Find all integers n and m, such that $nm \geq 9$, that satisfy the equation $2mn + n = 14$.

Solution Factoring the given equation gives us

$(2m+1)n=14$

Since $(2m+1)$ is an odd number, n must be even. This can be written as

1. $\begin{cases} 2m+1=1 \\ n=14 \end{cases} \Leftrightarrow m=0,\ n=14,\ nm=0$ not a solution

2. $\begin{cases} 2m+1=7 \\ n=2 \end{cases} \Leftrightarrow m=3,\ n=2,\ mn=6<9$ not a solution

3. $\begin{cases} 2m+1=-7 \\ n=-2 \end{cases} \Leftrightarrow m=-4,\ n=-2,\ nm=8<9$ not a solution

4. $\begin{cases} 2m+1=-1 \\ n=-14 \end{cases} \Leftrightarrow m=-1,\ n=-14,\ nm=14\geq 9$ true.

Answer $m=-1$ and $n=-14$.

In the following problem, we will learn how, by adding an expression that is equal to zero, we are able to complete a square or cube.

Problem 66 Assume that $a>1$ is the root of $x^3-x-1=0$. Evaluate $\sqrt[3]{3a^2-4a}+\sqrt[3]{3a^2+4a+2}$.

Solution Since $a^3-a-1=0$, by adding or subtracting it inside the radicals we will not change anything! We can then use the cube of difference and cube of sum formulas:

$$\sqrt[3]{3a^2-4a}=\sqrt[3]{3a^2-4a-(a^3-a-1)}=\sqrt[3]{1-3a+3a^2-a^3}=\sqrt[3]{(1-a)^3}=1-a$$

$$\sqrt[3]{3a^2+4a+2+(a^3-a-1)}=\sqrt[3]{(1+a)^3}=1+a$$

Therefore adding the left and the right sides, we get $\sqrt[3]{3a^2-4a}+\sqrt[3]{3a^2+4a+2}=1-a+1+a=2$.

Answer 2.

Figure 2.12 Pascal's triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

2.3.2 Newton's Binomial Theorem

You are probably familiar with Pascal's triangle (**Blaisé Pascal**, French mathematician, 1623–1662) that allows one to raise $(x+y)$ or $(x-y)$ to an integer power and find appropriate coefficients of the expansion (Figure 2.12).

Thus after making a Pascal triangle as above, and considering that the starting 1 belongs to a zero row and selecting, for example, row 5, we can evaluate the fifth power of $(x+y)$:

$$(x+y)^5 = 1\cdot x^5 + 5\cdot x^4 y + 10\cdot x^3 y^2 + 10\cdot x^2 y^3 + 5\cdot xy^4 + 1\cdot y^5$$

In this way you could find coefficients for any other power of $(x+y)$ or $(x-y)$. However, it is not a very efficient way if the power is greater than 10. In order to expand any power of $(x+y)$, we will introduce Newton's Binomial Theorem:

$$\begin{aligned}(u+v)^n &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} u^{n-k} v^k \\ &= u^n + nu^{n-1}v + \frac{n(n-1)}{2} u^{n-2} v^2 + \ldots + nuv^{n-1} + v^n\end{aligned} \tag{2.23}$$

This identity is known as *Newton's Binomial Theorem* because it was first stated by **Isaac Newton**, and $(x+y)$ is a "binomial," an expression with two terms.

The binomial coefficient in the formula above is often denoted by the symbol $C_n^k = \binom{n}{k}$ (pronounced "n choose k") and defined for nonnegative integers n and k by formulas

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad k = 0, 1, \ldots, n. \tag{2.24}$$

Here and below we introduce the **factorial** of a nonnegative integer n, denoted by $n\,!$ that is the product of all positive integers less than or equal to n.

The factorial can be written as

$$n! = 1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)\cdot n$$

Zero factorial equals one ($0! = 1$) and $1! = 1$. We can continue as follows:

$$\begin{aligned}
&2! = 1 \cdot 2 = 2\\
&3! = 1 \cdot 2 \cdot 3 = 2! \cdot 3 = 6\\
&4! = 1 \cdot 2 \cdot 3 \cdot 4 = 3! \cdot 4 = 6 \cdot 4 = 24\\
&5! = 4! \cdot 5 = 24 \cdot 5 = 120\\
&\qquad\qquad \ldots\\
&n! = (n-1)!n
\end{aligned}$$

Each time we use formula (2.24) we try to factor the factorial in the numerator so that one of the factors would match with a factorial in the denominator. Because $\frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!}$, we also can state the following property of the binomial coefficients:

$$C_n^k = C_n^{n-k} \tag{2.25}$$

By applying formula (2.24) directly, it is easy to show that

$$C_n^{n-k} = \binom{n}{n-k} = C_n^k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Let us evaluate some of the binomial coefficients:

$$\begin{aligned}
C_n^1 &= \frac{n!}{1!(n-1)!} = \frac{(n-1)!n}{(n-1)!} = n\\
C_n^2 &= \frac{n!}{2!(n-2)!} = \frac{(n-2)!(n-1)n}{2(n-2)!} = \frac{(n-1)n}{2}\\
C_n^{n-1} &= \frac{n!}{(n-1)!(n-(n-1))!} = \frac{(n-1)!n}{(n-1)! \cdot 1} = n
\end{aligned}$$

If you have taken a probability course or even introductory statistics, then you are familiar with binomial coefficients and their properties. In statistics, when someone selects a simple random sample of size n from a population of size N, the total number of possible samples can be found using the formula

$$C_N^n = \binom{N}{n} = \frac{N!}{n!(N-n)!}$$

Formula (2.23) can also be rewritten for the nth power of the difference of u and v, by placing “minus” before each odd power of v. For example, if $v = 1$ and $n = 100$, we obtain the following:

$$(u-1)^{100} = u^{100} - 100u^{99} + \frac{100 \cdot 99}{2} u^{98} - \ldots - 100u + 1$$

Problem 67 Show that

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}, k = 1, 2, \ldots n \tag{2.26}$$

Proof The proof can be done directly by applying formula (2.24):

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{(k)!(n-k)!} = \frac{n!k + n!(n-k+1)}{(k)!(n-k+1)!}$$
$$= \frac{n!(n+1)}{(k)!(n-k+1)!} = \frac{(n+1)!}{(k)!(n+1-k)!} = \binom{n+1}{k}$$

Next, let us show that Newton's formula is true.

Theorem 21 For any nonnegative number n, the following statement is valid:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \tag{2.27}$$

This statement can be deduced to

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \quad \text{for any } a, b. \tag{2.28}$$

Proof We will conduct the proof of (2.27) by induction.

1. Let $n = 1$. $(1+x)^1 = \sum_{k=0}^{1} \binom{1}{k} x^k = \binom{1}{0} x^0 + \binom{1}{1} x^1 = 1 + x$.
2. Assume that (2.27) is true for $n = t$, $t \geq 1$, i.e.,

$$(1+x)^t = \sum_{k=0}^{t} \binom{t}{k} x^t$$
$$= \binom{t}{0} + \binom{t}{1} x + \binom{t}{2} x^2 + \ldots + \binom{t}{t-1} x^{t-1} + \binom{t}{t} x^t \tag{2.29}$$

3. Let us show that (2.27) is also true for $n = t + 1$, i.e., $(1+x)^{t+1} = \sum_{k=0}^{t=1} \binom{t+1}{k} x^k$

$$
\begin{aligned}
(1+x)^{t+1} &= (1+x)(1+x)^t \\
&= (1+x)\left[\sum_{k=0}^{t}\binom{t}{k}x^t\right] \\
&= (1+x)\cdot\left[\binom{t}{0}+\binom{t}{1}x+\binom{t}{2}x^2+\ldots+\binom{t}{t-1}x^{t-1}+\binom{t}{t}x^t\right]
\end{aligned}
$$

Above we substituted (2.29) into the formula because we have assumed it to be true. Using distributive law we will obtain the following:

$$
\begin{aligned}
(1+x)^{t+1} = &\binom{t}{0}+\binom{t}{1}x+\binom{t}{2}x^2+\ldots+\binom{t}{t-1}x^{t-1}+\binom{t}{t}x^t \\
&+\binom{t}{0}x+\binom{t}{1}x^2+\binom{t}{2}x^3+\ldots+\binom{t}{t-1}x^t+\binom{t}{t}x^{t+1}.
\end{aligned}
$$

Next, we will combine like terms and use the property of binomial coefficients such as (2.26) and the formulas below:

$$
\begin{aligned}
C_n^0 &= \binom{n}{0} = C_{n+1}^0 = \binom{n+1}{0} = C_m^0 = \binom{m}{0} = 1 \\
C_n^n &= \binom{n}{n} = C_{n+1}^{n+1} = \binom{n+1}{n+1} = C_m^m = \binom{m}{m} = 1
\end{aligned}
$$

$$
\begin{aligned}
(1+x)^{t+1} &= \binom{t+1}{0}+\left(\binom{t}{0}+\binom{t}{1}\right)x+\left(\binom{t}{1}+\binom{t}{2}\right)x^2+\ldots \\
&\quad+\left(\binom{t}{t-1}+\binom{t}{t}\right)x^t+\binom{t+1}{t+1}x^{t+1} \\
&= \binom{t+1}{0}+\binom{t+1}{1}x+\binom{t+1}{2}x^2+\ldots+\binom{t+1}{t+1}x^{t+1} \\
&= \sum_{k=0}^{t}\binom{t+1}{k}x^k.
\end{aligned}
$$

Because we have shown it to be true for $n = t+1$, by induction we have proven that it is true for all $n \in N$.

In order to prove (2.28), we will use a similar induction approach, but because we understand now how to proceed, we will use only sigma notation.

Problem 68 Prove that $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$.

Proof

1. For $n = 1$ the statement is true (please check yourself).
2. Assume that $A(n)$ is true and that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.
3. Let us show that $A(n+1)$ is also true:

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b)\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$$
$$= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k+1}$$

Next, we will do a little "trick" by introducing a new variable $j = k+1$. Then $k = j-1$, and if k is changing between 0 and n, then j will change between 1 and $n+1$. Now the summation in our two sums will use this new index j:

$$= \sum_{j=1}^{n+1}\binom{n}{j-1}a^j b^{n+1-j} + \sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j+1}$$
$$= \sum_{j=1}^{n+1}\binom{n}{j-1}a^j b^{n+1-j} + \sum_{j=0}^{n}\binom{n}{j}a^j b^{n+1-j}$$

Combining like terms $a^m b^{n+1-m}$ from $j = 1$ to $j = n$ and applying formula (2.26) again we complete the proof:

$$= \sum_{j=1}^{n}\left(\binom{n}{j-1} + \binom{n}{j}\right)a^j b^{n+1-j} + \binom{n}{0}a^0 b^{n+1-0} + \binom{n}{n}a^{n+1}b^{n+1-(n+1)}$$
$$= \sum_{j=1}^{n}\binom{n+1}{j}a^j b^{n+1-j} + \binom{n+1}{0}a^0 b^{n+1-0} + \binom{n+1}{n+1}a^{n+1}b^{n+1-(n+1)}$$
$$= \sum_{j=0}^{n+1}\binom{n+1}{j}a^j b^{n+1-j} = (a+b)^{n+1}.$$

The proof is completed.

Next, we will discuss Pascal's triangle and other properties of the binomial coefficients.

Let us expand (2.28):

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \ldots + \frac{n(n-1)}{2}a^2 b^{n-2} + nab^{n-1} + b^n \tag{2.30}$$

We notice that the coefficients of the terms are symmetric with respect to the "center" of the formula. Thus, going from the ends to the center of the formula, we have 1 and 1, n and n, etc. This symmetry follows from (2.25).

In high school, many students learn Pascal's triangle, which has already been mentioned above. Using Figure 2.12 let us see how some properties of the binomial coefficients can be visualized from it.

1. The numbers in each row are the coefficients of $(x + y)^n$.
2. In each row, the numbers symmetric with respect to the center of the row are the same. This is true because $C_n^k = C_n^{n-k}$.
3. The numbers in each consecutive row can be obtained from the previous one by the rule $C_n^k + C_n^{k+1} = C_{n+1}^{k+1}$. Thus number 10 in row 5 is the sum of 4 and 6 in the previous row 4.

Lemma 1 *The sum of all binomial coefficients in the row corresponding to power n equals* 2^n.

Proof Thus, if $n = 1$ we obtain $1 + 1 = 2$, if $n = 2$ we have $1 + 2 + 1 = 4 = 2^2$, for $n = 3$ we get $1 + 3 + 3 + 1 = 8 = 2^3$, etc.

Let us prove these properties of the binomial coefficients, i.e.,

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n. \tag{2.31}$$

The proof of this statement is simple.

Consider formula (2.27) again:

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

If this formula is correct, then it is correct for any value of x. Let x be 1, then the answer follows immediately:

$$(1 + 1)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Remark Though Newton's binomial formula can be used for any power of n, it is useful to memorize some often-used formulas without deriving them all the time; no need to reinvent the wheel each time you need one. This is why some rules and formulas I recommend remembering by heart.

Problem 69 Find m and n if the following is true:

$$\binom{n+1}{m+1} : \binom{n+1}{m} : \binom{n+1}{m-1} = 5 : 5 : 3$$

Solution On one hand,

$$\binom{n+1}{m+1} : \binom{n+1}{m} = \frac{(n+1)!}{(m+1)!(n-m)!} \cdot \frac{m! \cdot (n-m+1)!}{(n+1)!} = \frac{n-m+1}{m+1} = \frac{5}{5} = 1.$$

On the other hand,

$$\binom{n+1}{m} : \binom{n+1}{m-1} = \frac{(n+1)!}{(m)!(n-m+1)!} \cdot \frac{(m-1)! \cdot (n-m+2)!}{(n+1)!} = \frac{n-m+2}{m} = \frac{5}{3}.$$

Answer $n = 6$, $m = 3$.

Problem 70 Evaluate the constant term of the expansion: $\left(1 + x + \frac{1}{x}\right)^8$.

Solution Let $u = x + \frac{1}{x}$, then the given formula can be rewritten as

$$(1+u)^8 = 1 + 8u + \binom{8}{2}u^2 + \binom{8}{3}u^3 + \ldots + \binom{8}{7}u^7 + u^8, \text{ where}$$

$$u^k = \left(x + \frac{1}{x}\right)^k = x^k + kx^{k-2} + \ldots + \binom{k}{m}x^{k-2m} + \ldots + \frac{1}{x^k}.$$

We can see that u^k will have zero power whenever $k = 2m$. There are four possible pairs when this is true for k and m: $(k, m) : \{(2,1), (4,2), (6,3), (8,4)\}$.

Thus the constant term is

$$1 + \binom{8}{2} \cdot \binom{2}{1} + \binom{8}{4} \cdot \binom{4}{2} + \binom{8}{6} \cdot \binom{6}{3} + \binom{8}{8}\binom{8}{4}$$
$$= 1 + 56 + 420 + 560 + 70 = 1107.$$

Answer 1107.

2.4 Biquadratic Equations: Special Substitutions

Some equations can be simplified to a quadratic type equation if we introduce a new variable.

Example 1 Solve $4x^6 - 13x^3 + 9 = 0$.

By introducing a new variable $y = x^3$ $\left(x = \sqrt[3]{y}\right)$, the equation will become a quadratic in y:

$$4y^2 - 13y + 9 = 0$$
$$y_1 = 1, y_2 = \frac{9}{4}$$
$$x_1 = 1, x_2 = \sqrt[3]{\frac{9}{4}}$$

Example 2 Solve $15x^{\frac{2}{3}} + 11x^{\frac{1}{3}} - 26 = 0$.

Denote $y = x^{\frac{1}{3}}$, $x > 0$ $(x = y^3)$.

$$15y^2 + 11y - 26 = 0$$
$$y_1 = 1, y_2 = -\frac{26}{15}.$$
$$x = 1$$

Here we had to remember that rational exponent function $y = x^{\frac{1}{3}}$ is defined only for positive values of the independent variable. This is why we have one answer, $x = 1$.

For some equations, a good substitution can be not that obvious and we need to learn how to recognize it. For example, if an equation contains the terms like $\frac{a}{x}; \frac{x}{b}$ raised to some power, you can try to introduce a new variable as

$$y = \frac{a}{x} \pm \frac{x}{b}. \tag{2.32}$$

Then for example,

$$y^2 = \left(\frac{a}{x} \pm \frac{x}{b}\right)^2 = \frac{a^2}{x^2} \pm 2\,\frac{a}{b} + \frac{x^2}{b^2}$$
$$\frac{a^2}{x^2} + \frac{x^2}{b^2} = y^2 \mp 2\,\frac{a}{b}$$

Problem 71 Solve the equation $\frac{x^2}{3} + \frac{48}{x^2} = 10\left(\frac{x}{3} - \frac{4}{x}\right)$.

Solution We can try to introduce a new variable using (2.32):

$$y = \frac{x}{3} - \frac{4}{x} \tag{2.33}$$

Squaring both sides of the equation above we obtain
$y^2 = \left(\frac{x}{3} - \frac{4}{x}\right)^2 = \frac{x^2}{9} - \frac{8}{3} + \frac{16}{x^2}$, which can be written as

$$y^2 + \frac{8}{3} = \frac{x^2}{9} + \frac{16}{x^2}$$

Multiplying both sides by 3 we have

$$3y^2 + 8 = \frac{x^2}{3} + \frac{48}{x^2}$$

This we can substitute back into the original equation and get the following:

$$3y^2 + 8 = 10y \text{ or } 3y^2 - 10y + 8 = 0$$

This quadratic equation in variable y can be solved easily and it has the following roots: $y_1 = 2$ and $y_2 = \frac{4}{3}$

Next, we have to solve (2.33) for each y value.

$$1.\ \frac{x}{3} - \frac{4}{x} = 2. \text{ and } 2.\ \frac{x}{3} - \frac{4}{x} = \frac{4}{3}.$$

The first equation is equivalent to a quadratic equation:

$$\begin{aligned}
&x^2 - 6x - 12 = 0, \quad x \neq 0 \\
&x_{1,2} = 3 \pm \sqrt{9 + 12} \\
&x_1 = 3 + \sqrt{21} \\
&x_2 = 3 - \sqrt{21}
\end{aligned}$$

The second equation is equivalent to a quadratic equation:

$$\begin{aligned}
&x^2 - 4x - 12 = 0, \quad x \neq 0 \\
&x_{3,4} = 2 \pm \sqrt{4 + 12} \\
&x_3 = 6 \\
&x_4 = -2
\end{aligned}$$

Therefore, the original equation has four real roots.

Answer $x = 3 + \sqrt{21},\ x = 3 - \sqrt{21},\ x = 6,\ x = -2.$

Remark You can argue that the problem could be solved without any substitution if we multiply both sides of the equation by $3x^2$. Then the given equation would take the following polynomial form of the fourth degree $x^4 - 10x^3 + 120x + 144 = 0$, and fortunately, by the Rational Zero Theorem, we could find two integer solutions $x = -2$ and $x = 6$. Then by dividing the given polynomial by $(x+2)$ and $(x-6)$ we would obtain a quadratic equation that would give us two other irrational roots. I agree that the method introduced in this section is not the only method of solving the equation given in this particular problem. However, it is always nice to solve a problem in several ways. In my opinion, learning how some equations can be simplified is important for developing problem solving skills. Indeed, the more you know, the better you will be prepared for any math contest. For example, a similar idea of substitution can be applied to some cubic equations and for solving symmetric polynomial equations. You will see it soon in the text.

Other types of substitution can be seen in the following problems.

Problem 72 Solve the equation $\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1$.

Solution Let us introduce a new variable:

$y = \sqrt{x-1} \geq 0, x \geq 1$. Then $y^2 = x - 1$, and $x = y^2 + 1$.

In terms of the new variable, our equation looks like

$$\sqrt{y^2+1+3-4y} + \sqrt{y^2+1+8-6y} = 1$$
$$\sqrt{(y-2)^2} + \sqrt{(y-3)^2} = 1$$

Next, we will open each square root as an absolute value:

$$|y-2| + |y-3| = 1. \tag{2.34}$$

We will solve this equation using the method of intervals. Let us place 2 and 3 on the number line in increasing order. They divide the number line into three intervals: $y < 2, 2 \leq y < 3, y \geq 3$. We will solve the modulus equation on each of these intervals. Thus

1.

$$y < 2.$$
$$2 - y + 3 - y = 1$$
$$y = 2$$
$$\varnothing$$

2.

$$\begin{aligned}&2 \le y < 3\\&y - 2 + 3 - y = 1\\&1 = 1 \quad \text{(true)}\\&y \in [2, 3)\end{aligned}$$

3.

$$\begin{aligned}&y \ge 3\\&y - 2 + y - 3 = 1\\&y = 3\end{aligned}$$

Therefore, (2.34) is true on the closed interval $y \in [2, 3]$.

Next, we will find the interval for x. The following is valid:

$$\begin{aligned}&2 \le \sqrt{x-1} \le 3\\&4 \le x - 1 \le 9\\&5 \le x \le 10.\end{aligned}$$

Answer $x \in [5, 10]$.

We obtained a very interesting case, because the solution to the equation is the closed interval and then any real value of x from that interval makes the equation true. You can check for example $x = 6$. We need to show that

$$\sqrt{6 + 3 - 4\sqrt{5}} + \sqrt{6 + 8 - 6\sqrt{5}} = 1$$

or that

$$\sqrt{9 - 4\sqrt{5}} + \sqrt{14 - 6\sqrt{5}} = 1. \tag{2.35}$$

At first glance, it does not look right. However, let us complete the square under each radical, and consider

$$\begin{aligned}\left(2 - \sqrt{5}\right)^2 &= 2^2 - 2 \cdot 2 \cdot \sqrt{5} + \left(\sqrt{5}\right)^2 = 9 - 4\sqrt{5}\\ \left(3 - \sqrt{5}\right)^2 &= 3^2 - 2 \cdot 3 \cdot \sqrt{5} + \left(\sqrt{5}\right)^2 = 14 - 6\sqrt{5}\end{aligned}$$

Because $\sqrt{5} > 2$ and $\sqrt{5} < 3$, finally, (2.35) becomes $\left|2 - \sqrt{5}\right| + \left|3 - \sqrt{5}\right| = \sqrt{5} - 2 + 3 - \sqrt{5} = 1$.

A similar result can be obtained for any real $x \in [5, 10]$.

Problem 73 Solve the equation $\sqrt{\dfrac{x+1}{x-1}} - \sqrt{\dfrac{x-1}{x+1}} = \dfrac{3}{2}$.

Solution Instead of working with radicals, we can denote

$$y = \sqrt{\frac{x+1}{x-1}} \Rightarrow \frac{1}{y} = \sqrt{\frac{x-1}{x+1}}.$$

Then the equation can be rewritten as

$$y - \tfrac{1}{y} = \tfrac{3}{2} \text{ or } 2y^2 - 2 = 3y, y > 0.$$

With the solution $y = 2$. (Note that the second root of the quadratic equation, $y = -1/2$, would not satisfy the original equation domain.)

Next, we will find the corresponding x:

$$\begin{aligned}
&\sqrt{\frac{x+1}{x-1}} = 2 \\
&\frac{x+1}{x-1} = 4 \\
&x + 1 = 4x - 4 \\
&3x = 5 \\
&x = \frac{5}{3}
\end{aligned}$$

Answer $x = \frac{5}{3}$.

Problem 74 Solve the equation $(x^2 + x - 1) \cdot (x^2 + x + 1) = 2$.

Solution You can simplify this equation of the 4th degree to a quadratic by making a substitution of $t = x^2 + x$

Then the equation becomes

$$\begin{aligned}
&(t-1)(t+1) = 2 \\
&t^2 - 1 = 2 \\
&t_1 = \sqrt{3}, t_2 = -\sqrt{3} \\
&x^2 + x = \sqrt{3} \text{ or } x^2 + x = -\sqrt{3}
\end{aligned}$$

Only the first equation leads to real solutions in x:

$$\begin{aligned}
&x^2 + x - \sqrt{3} = 0 \\
&x = \frac{-1 + \sqrt{1 + 4\sqrt{3}}}{2} \text{ or } x = \frac{-1 - \sqrt{1 + 4\sqrt{3}}}{2}
\end{aligned}$$

Answer $x_{1,2} = \frac{-1 \pm \sqrt{1+4\sqrt{3}}}{2}$.

Remark If you tried to solve this equation directly without recognizing an appropriate substitution you would get the following polynomial equations of the fourth degree: $x^4 + 2x^3 + x^2 - 3 = 0$.

You would see that there are no integer solutions to this equation because numbers 1, -1, 3, and -3 do not make this polynomial zero. However, if you rewrite it as $x^4 + 2x^3 + x^2 = 3$, you could notice a trinomial square on the left and then the solution is similar to the previous one:

$$\begin{aligned}&(x^2 + x)^2 = 3\\ &x^2 + x = \pm\sqrt{3},\ \text{etc.}\end{aligned}$$

Problem 75 Solve the equation $4^x + 2^{x+2} + 7 = p - 4^{-x} - 2 \cdot 2^{1-x}$.

Solution Let us rewrite this equation in the following form:

$$4^x + \frac{1}{4^x} + 4 \cdot \left(2^x + \frac{1}{2^x}\right) + 7 - p = 0$$

Denote

$$y = 2^x > 0$$

Then $y^2 = 4^x$, $\frac{1}{y} = 2^{-x}$,, etc.

In terms of the new variables our equation becomes

$$y^2 + \frac{1}{y^2} + 4 \cdot \left(y + \frac{1}{y}\right) + 7 - p = 0 \tag{2.36}$$

It is natural to introduce another variable

$$z = y + \frac{1}{y} \tag{2.37}$$

from which

$$\begin{aligned}z^2 &= y^2 + 2 + \frac{1}{y^2}\\ y^2 + \frac{1}{y^2} &= z^2 - 2\end{aligned}$$

Now instead of solving a polynomial equation of 4th degree in variable x (2.36), we will solve a quadratic in z:

$$z^2 + 4z + (5 - p) = 0.$$

Because the coefficient of z is even, we will use D/4 formula (2.3) for the roots of a quadratic equation. The value of x is real if D/4 is nonnegative:

$$\frac{D}{4} = 2^2 - (5 - p) = p - 1 \geq 0$$
$$p \geq 1$$

And the roots are

$$z = -2 \pm \sqrt{p - 1}.$$

Because z is positive as the sum of two positive expressions, then out of two possible values of z, we will select only one, such as

$$z = \sqrt{p - 1} - 2, \quad p > 5.$$

Substituting it into (2.37), we obtain a new quadratic equation in y:

$$y^2 - \left(\sqrt{p - 1} - 2\right) \cdot y + 1 = 0 \tag{2.38}$$

Consider the discriminant for (2.38):

$$D = (\sqrt{p - 1} - 2)^2 - 4 \geq 0$$
$$p - 1 - 4\sqrt{p - 1} + 4 - 4 \geq 0$$
$$p - 1 \geq 4\sqrt{p - 1}$$
$$(p - 1)^2 \geq 16(p - 1)$$
$$(p - 1)(p - 17) \geq 0$$

Because the first factor is positive, in order for the inequality to be true, $p - 17 \geq 0$ must hold or

$$p \geq 17. \tag{2.39}$$

Do you remember that we obtained this restriction on p by solving Problem 30 of Chapter 1? From (2.38) and under condition (2.39), we obtain two solutions for variable y:

$$y_1 = \frac{\sqrt{p - 1} - 2 + \sqrt{p - 1 - 4\sqrt{p - 1}}}{2}$$
$$y_2 = \frac{\sqrt{p - 1} - 2 - \sqrt{p - 1 - 4\sqrt{p - 1}}}{2}$$

Both values of y satisfying the inequality (2.39) are positive at any p.

The corresponding values of x can be found as

$$x_1 = \log_2\left(\frac{\sqrt{p-1} - 2 + \sqrt{p-1-4\sqrt{p-1}}}{2}\right)$$
$$x_2 = \log_2\left(\frac{\sqrt{p-1} - 2 - \sqrt{p-1-4\sqrt{p-1}}}{2}\right)$$

For example, if $p = 17$, we obtain that $x_1 = x_2 = 0$.

2.5 Symmetric (Recurrent) Polynomial Equations

An equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ is called ***recurrent*** if and only if

$$a_k = a_{n-k}, \quad k = 0, 1, 2, \ldots, n. \tag{2.40}$$

We will consider separately recurrent equations of odd and even degree.

2.5.1 Symmetric Polynomial Equations of Even Degree

If for any polynomial of even degree written in descending form, the left and right coefficients are symmetric (equal), satisfying (2.40), such equation can be simplified by substitution:

$$y = x + \frac{1}{x} \tag{2.41}$$

Example Consider a polynomial equation of the 4th order:

$$ax^4 + bx^3 + cx^2 + bx + a = 0.$$

First, we will divide both sides of it by x^2 $(x \neq 0)$ and obtain its equivalent form:

$$ax^2 + bx + c + \frac{b}{x} + \frac{a}{x^2} = 0$$

This equation will be reduced to a *biquadratic equation* in variable y given by formula (2.40).

Thus, $y^2 = x^2 + \frac{1}{x^2} + 2$ or

$$x^2 + \frac{1}{x^2} = y^2 - 2 \tag{2.42}$$

Regrouping the terms and making substitution (2.41) and (2.42), we obtain a quadratic equation in y that can be solved using standard methods:

$$\begin{aligned} &a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0 \\ &a(y^2 - 2) + by + c = 0 \\ &ay^2 + by + (c - 2a) = 0. \end{aligned}$$

Problem 76 Solve the equation $x^4 - 5x^3 + 6x^2 - 5x + 1 = 0$.

Solution This equation is a symmetric polynomial equation of the 4th order, so we can use the method and substitution discussed above.

Dividing by the square of x, we obtain the new form of the equation:

$$x^2 - 5x + 6 - \frac{5}{x} + \frac{1}{x^2} = 0.$$

$$y = x + \frac{1}{x} \tag{2.43}$$

and

$$x^2 + \frac{1}{x^2} = y^2 - 2$$

$$\begin{aligned} &y^2 - 5y + 4 = 0 \\ &y_1 = 1, y_2 = 4. \end{aligned}$$

Next, for each value of y we have to solve (2.43):

1. $x + \frac{1}{x} = 1$
   $x^2 - x + 1 = 0$
   no real solutions

2. $x + \frac{1}{x} = 4$
   $x^2 - 4x + 1 = 0$
   $x = 2 + \sqrt{3}, \quad x = 2 - \sqrt{3}.$

Remark As you could see, for this problem the Rational Zero Theorem would not be helpful because the equation has two irrational roots. Our method allowed us to find these roots analytically. The graph of the polynomial function with two irrational roots is given in Figure 2.13.

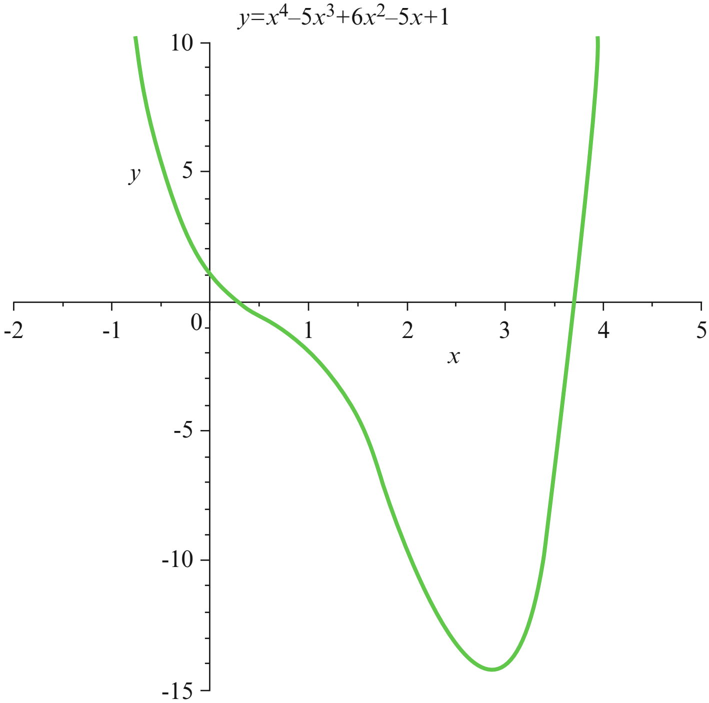


Figure 2.13 Sketch for Problem 76

Problem 77 Solve $4x^2 + 12x + \frac{12}{x} + \frac{4}{x^2} = 47$.

Solution Method 1: This equation is not polynomial but can be solved using a similar substitution:

$$y = 2x + \frac{2}{x}$$
$$y^2 = \left(4x^2 + \frac{4}{x^2}\right) + 8$$

Then the given equation will be reduced to a quadratic in y:

$$y^2 + 6y - 55 = 0$$
$$y_1 = -11, y_2 = 5$$

$$\begin{aligned} &y = -11 && y = 5 \\ &2x^2 + 11x + 2 = 0 && 2x^2 - 5x + 2 = 0 \\ &x_{1,2} = \frac{-11 \pm \sqrt{105}}{4} && x_3 = 2, x_4 = \frac{1}{2} \end{aligned}$$

Method 2: We can multiply both sides by x^2 and obtain polynomial equation of 4th order:

$$4x^4 + 12x^3 - 47x^2 + 12x + 4 = 0$$

Using the Rational Zero Theorem we can find that $x = 2, x = \frac{1}{2}$ are the roots, and then using Horner's algorithm we will divide the original polynomial by factors $(x - 2)$ and $\left(x - \frac{1}{2}\right)$ and obtain the quadratic equation $2x^2 + 11x + 2 = 0$ whose solution will give us the remaining roots.

Answer $\frac{-11 \pm \sqrt{105}}{4}; \frac{1}{2}; 2.$

2.5.2 *Symmetric Polynomial Equations of Odd Degree*

Lemma 2 *Any symmetric polynomial equation of odd degree has a solution* $x = -1$.

This lemma can be easily proven in general. Instead, let us consider a cubic symmetric equation:

$$ax^3 + bx^2 + bx + a = 0.$$

Regrouping the terms and factoring out common factors we obtain

$$\begin{aligned} &a(x^3 + 1) + b(x^2 + x) = 0 \\ &a(x + 1)(x^2 - x + 1) + bx(x + 1) = 0 \\ &(x + 1)(ax^2 - (a - b)x + a) = 0 \end{aligned}$$

The solution of $x = -1$ is obtained.

Remark A similar approach can be used for *a skew symmetric odd equation*. For example, the equation $ax^3 + bx^2 - bx - a = 0$ can be rewritten as $(x - 1)(ax^2 + (a + b)x + a) = 0$ and will have at least one real zero $x = 1$.

2.6 Cubic Equations

In general, any cubic equation $ax^3 + bx^2 + cx + d = 0,\ a \neq 0$ has at least one real zero and at most three real zeroes. Even ancient Babylonians tried to solve cubic equations. We can try to find such zeroes using the Rational Zero Theorem. However, if there is no rational zero, then finding the solution to a cubic equation can be challenging. In this section, we discuss some helpful methods and derive important formulas.

2.6.1 Vieta's Theorem for Cubic Equations

In the previous sections, we formulated Vieta's Theorem for quadratic equations. This important theorem is also valid for polynomials of higher degree and it is very useful for cubic equations. For example, if x_1, x_2, x_3 are the roots of the polynomial equation

$$x^3 + ax^2 + bx + c = 0 \tag{2.44}$$

then they satisfy the following system of equations:

$$\begin{cases} x_1 + x_2 + x_3 = -a \\ x_1 \cdot x_2 + x_2 \cdot x_3 + x_1 \cdot x_3 = b \\ x_1x_2x_3 = -c \end{cases} \tag{2.45}$$

The reverse theorem is also valid: Solutions of system (2.45) are the roots of cubic equation (2.44).

Remark In general, for any equation $ax^3 + bx^2 + cx + d = 0, a \neq 0$, the Vieta's Theorem can be applied. For example, if x_1, x_2, x_3 are the roots of the polynomial equation

$$ax^3 + bx^2 + cx + d = 0,\ a \neq 0 \tag{2.46}$$

then they satisfy the following system of equations:

$$\begin{cases} x_1 + x_2 + x_3 = -\dfrac{b}{a} \\ x_1 \cdot x_2 + x_2 \cdot x_3 + x_1 \cdot x_3 = \dfrac{c}{a} \\ x_1x_2x_3 = -\dfrac{d}{a} \end{cases} \tag{2.47}$$

Let us see how this can be used for solving the following problem.

Problem 78 Find such pairs of positive values of parameters a and c for which all three roots of the equation $2x^3 - 2ax^2 + (a^2 - 81)x - c = 0$ are natural numbers.

Solution Assume that we know all natural zeroes of the given equation, $x_1 \le x_2 \le x_3$; then they must satisfy Vieta's Theorem (2.45):

$$\begin{cases} x_1 + x_2 + x_3 = a \\ x_1 \cdot x_2 + x_2 \cdot x_3 + x_1 \cdot x_3 = \dfrac{a^2 - 81}{2} \\ x_1 x_2 x_3 = \dfrac{c}{2} \end{cases}$$

Using the true equality

$$(x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2(x_1 \cdot x_2 + x_2 \cdot x_3 + x_1 \cdot x_3)$$

the first two equations of the system can be combined as

$$(a)^2 = x_1^2 + x_2^2 + x_3^2 + 2 \cdot \frac{(a^2 - 81)}{2}$$
$$x_1^2 + x_2^2 + x_3^2 = 81$$

We can find the following triples of natural numbers, the sum of the squares of which is 81:

$$(x_1, x_2, x_3) = \{(1, 4, 8), (4, 4, 7), (3, 6, 6)\}$$

And for each triple we obtain the corresponding values of $a = x_1 + x_2 + x_3$ and corresponding value of parameter $c = 2 \cdot x_1 x_2 x_3$:

$$(1, 4, 8) : a = x_1 + x_2 + x_3 = 1 + 4 + 8 = 13,\ c = 2 \cdot x_1 x_2 x_3 = 2 \cdot 1 \cdot 4 \cdot 8 = 64$$
$$(4, 4, 7) : a = x_1 + x_2 + x_3 = 4 + 4 + 7 = 15,\ c = 2 \cdot x_1 x_2 x_3 = 2 \cdot 4 \cdot 4 \cdot 7 = 224$$
$$(3, 6, 6) : a = x_1 + x_2 + x_3 = 3 + 6 + 6 = 15,\ c = 2 \cdot x_1 x_2 x_3 = 2 \cdot 3 \cdot 6 \cdot 6 = 216$$

Finally, the positive pairs of the parameters are given as

$$(a, c) : \{(13, 64), (15, 224), (15, 216)\}.$$

Remark Because we obtained three different polynomial functions with all even coefficients, the polynomials can be simplified as

$$p_1(x) = x^3 - 13x^2 + 44x - 32 \Leftrightarrow (x_1, x_2, x_3) = (1, 4, 8)$$
$$p_2(x) = x^3 - 15x^2 + 72x - 112 \Leftrightarrow (x_1, x_2, x_3) = (4, 4, 7)$$
$$p_3(x) = x^3 - 15x^2 + 72x - 108 \Leftrightarrow (x_1, x_2, x_3) = (3, 6, 6)$$

2.6.2 The Babylonian Approach to Cubic Equations

It is known that ancient Babylonians could solve simple cubic equations (Table 2.1).

Table 2.1 Babylonian Method

N	n^3	n^2	$n^3 + n^2$
1	1	1	2
2	8	4	12
3	27	9	36
4	64	16	80
5	125	25	150
6	216	36	252
7	343	49	392
8	512	64	576
9	729	81	810
10	1000	100	1100

We will review their methods by solving the following problem.

Problem 79 A Babylonian tablet has been discovered that gives the value of $n^3 + n^2$ for $n = 1$ to 30. Make such a table for $n = 1$ to 10. Using the table solve the equation $x^3 + 2x^2 - 3136 = 0$.

Solution The equation $x^3 + 2x^2 - 3136 = 0$ is not ready to be solved using the table. Notice that the Table 2.1 gives us the sum of the cube and square of a number with the unit coefficients of each term. Let us divide the given equation by $2^3 = 8$ and then introduce a new variable $n = \frac{x}{2}$:

$$\frac{x^3}{2^3} + 2 \cdot \frac{x^2}{2^3} = \frac{3136}{2^3}$$
$$\left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^2 = 392$$
$$n^3 + n^2 = 392$$
$$n = 7$$
$$x = 14.$$

Answer $x = 14$.

Problem 80 Solve a cubic equation $3x^3 + 5x = 16$.

Solution First, this equation can be rewritten as $f(x) = 3x^3 + 5x - 16 = 0$ and before we find any way to solve this equation, we know that the function on the left is monotonically increasing as the sum of two increasing functions, and hence it will have only one real root. Second, it does not have a quadratic term (linear instead), so it will be difficult to use the Babylonian table here. It looks like a good idea to multiply this by 9 and then rewrite the equation as

$$\begin{aligned}&(3x)^3 + 15 \cdot 3x = 144\\&y^3 + 15y - 144 = 0,\ y = 3x\end{aligned} \tag{2.48}$$

Do not try to solve this cubic equation using the quadratic formula because it works only for quadratic equations. Because the Rational Zero Theorem does not give any help (this equation does not have any integer solutions either), one technique is to apply Cardano's formula that gives a closed form solution for an equation of the form $x^3 + px + q = 0$. This formula is derived later in the text.

The other option here is to try the substitution:

$$y = \frac{5}{z} - z \tag{2.49}$$

from which we obtain that

$$z^2 + y \cdot z - 5 = 0 \tag{2.50}$$

From the Vieta's Theorem for a quadratic equation we have that

$$y = -(z_1 + z_2), \tag{2.51}$$

where z_1, z_2 are the solutions of the quadratic equation (2.50).

Raise both sides of (2.49) to the third power:

$$y^3 = \left(\frac{5}{z} - z\right)^3 = \frac{125}{z^3} - 3 \cdot \frac{25}{z} + 3 \cdot 5z - z^3 \tag{2.52}$$

Next, we will evaluate $15y$:

$$15y = 15\left(\frac{5}{z} - z\right) = \frac{75}{z} - 15z \tag{2.53}$$

Adding (2.52) and (2.53) and substituting the result in (2.48), we obtain

$$y^3 + 15y = \frac{125}{z^3} - z^3 = 144$$

or

$$\frac{125}{z^3} - z^3 = 144 \tag{2.54}$$

Equation (2.54) becomes quadratic in a new variable

$$t = z^3 \tag{2.55}$$

and can be written as

$$t^2 + 144t - 125 = 0$$
$$t_1 = -72 + \sqrt{5309} \quad \text{and} \quad t_2 = -72 - \sqrt{5309}$$

Then using (2.55) we will evaluate two values of z:

$$z_1 = \sqrt[3]{-72 + \sqrt{5309}} \approx 4.16 > 0 \quad z_2$$
$$= \sqrt[3]{-72 - \sqrt{5309}} = -\sqrt[3]{72 + \sqrt{5309}} \approx -5.25 < 0 \tag{2.56}$$

For each z using (2.49) we can obtain the corresponding value of y:

$$\begin{aligned} y_1 &= -\sqrt[3]{-72 + \sqrt{5309}} + \frac{5}{\sqrt[3]{-72 + \sqrt{5309}}} \\ y_2 &= \sqrt[3]{72 + \sqrt{5309}} - \frac{5}{\sqrt[3]{72 + \sqrt{5309}}} \end{aligned} \tag{2.57}$$

Finally, dividing each y by 3 we will obtain two real x as

$$x_1 = \frac{y_1}{3}, \quad x_2 = \frac{y_2}{3}.$$

On the other hand, we could substitute (2.56) into (2.51) and evaluate just one value of y as

$$y = -\sqrt[3]{-72 + \sqrt{5309}} + \sqrt[3]{72 + \sqrt{5309}} \tag{2.58}$$

Then we would have just one value of x.

Question Are there two or one real zeros? Which formula (2.57) or (2.58) should we use?

Usually my students get only formulas (2.57) and answer that the polynomial equation has two real zeroes given by (2.57) and that it is fine because a polynomial function can have two X-intercepts. I say then that yes, a polynomial function of third degree can have one, two, or three X-intercepts, but we need to find out if the

given polynomial function of this special type can have two X-intercepts. I also tell them that independently of (2.57), they could get just one solution for y given by (2.58) and ask them to solve this "contradiction" because formulas (2.57) and (2.58) at first glance do not look the same.

Consider a polynomial function $p(x) = x^3 + 3ax - 2b,\ a > 0$.

Our polynomial function above fits this type at $a = 5$, $b = 72$.

Let us take its first derivative:

$$p'(x) = 3x^2 + 3a = 3\left(x^2 + a\right) > 0,\ \forall x \in R$$

Moreover, our polynomial has a negative Y-intercept $\left(p(0) = -2b < 0\right)$ and passing through this intercept, the function is monotonically increasing; then its X-intercept must be positive and unique!

Further, if we rationalize the denominator in the second terms of the formulas (2.57) and use the difference of squares formula (2.10), we would obtain that

$$\begin{aligned}\frac{-5}{\sqrt[3]{72+\sqrt{5309}}} &= \frac{-5\cdot\sqrt[3]{72-\sqrt{5309}}}{\sqrt[3]{\left(72+\sqrt{5309}\right)\left(72-\sqrt{5309}\right)}} \\ &= \frac{-5\cdot\sqrt[3]{72-\sqrt{5309}}}{\sqrt[3]{-125}} = \sqrt[3]{72-\sqrt{5309}}\end{aligned}$$

and that

$$\frac{5}{\sqrt[3]{-72+\sqrt{5309}}} = \frac{5\cdot\sqrt[3]{-72-\sqrt{5309}}}{\sqrt[3]{\left(-72+\sqrt{5309}\right)\left(-72-\sqrt{5309}\right)}} = \sqrt[3]{72+\sqrt{5309}}$$

Thus in the formulas (2.57) both values of y are identical and equal to the value for y given by (2.58).

Answer $x = \dfrac{-\sqrt[3]{-72+\sqrt{5309}}+\sqrt[3]{72+\sqrt{5309}}}{3}.$

Remark Of course, we do not need calculus in order to show the existence of only one real root for this equation. Consider $x^3 + px + q = 0, (p > 0, q \neq 0)$. If $p > 0$, then $f(x) = x^3 + px + q$ is a monotonically increasing function ($g(x) = -f(x)$ is monotonically decreasing) and it can have only one real zero!

2.6.3 Special Substitutions for Cubic Equations

Special Substitution for Cubic Equations of Type $x^3 + 3ax = 2b,\ \ (a > 0)$

As we have already learned by solving Problem 81, this equation we will have only one real zero. Let us try a new variable:

$$x = \frac{a}{y} - y \tag{2.59}$$

First, we will multiply both sides by y and obtain

$$a - y^2 = xy$$
$$y^2 + x \cdot y - a = 0$$

Applying Vieta's Theorem to the quadratic equation, we have that $y_1 \cdot y_2 = -a$ and that

$$x = -(y_1 + y_2) \tag{2.60}$$

Second, let us cube both sides:

$$x^3 = \left(\frac{a}{y} - y\right)^3$$
$$x^3 = \frac{a^3}{y^3} - \frac{3a^2}{y} + 3ay - y^3 \tag{2.61}$$

On the other hand, we can obtain

$$3ax = 3a\left(\frac{a}{y} - y\right) = \frac{3a^2}{y} - 3ay \tag{2.62}$$

Adding (2.61) and (2.62) we obtain that

$$x^3 + 3ax = \frac{a^3}{y^3} - y^3.$$

Now, the given equation can be written as

$$\frac{a^3}{y^3} - y^3 = 2b \tag{2.63}$$

I hope you recognized that (2.63) is a quadratic type equation that can be obtained by substitution:

$$z = y^3 \Rightarrow y = \sqrt[3]{z} \tag{2.64}$$

$$\begin{aligned} &z^2 + 2bz - a^3 = 0 \\ &z_1 = -b + \sqrt{b^2 + a^3} \\ &z_2 = -b - \sqrt{b^2 + a^3} \end{aligned} \tag{2.65}$$

For each z given by (2.65), the values of y can be found from (2.64) as a cubic root of z

$(y_1 = \sqrt[3]{-b+\sqrt{b^2+a^3}};\quad y_2 = \sqrt[3]{-b-\sqrt{b^2+a^3}})$ and then using (2.60), we find the corresponding x:

$$x = -\sqrt[3]{-b+\sqrt{b^2+a^3}} - \sqrt[3]{-b-\sqrt{b^2+a^3}} = \sqrt[3]{b+\sqrt{b^2+a^3}} - \sqrt[3]{-b+\sqrt{b^2+a^3}}$$

Answer $x = \sqrt[3]{b+\sqrt{b^2+a^3}} - \sqrt[3]{-b+\sqrt{b^2+a^3}}.$

Special Substitution for Cubic Equations of Type

$$x^3 - 3ax = 2b \quad (a > 0, b > 0)$$

Let us find out if we can do something similar to the (2.59) substitution here, such as

$$x = \frac{a}{y} + y \tag{2.66}$$

First, we will multiply both sides by y and obtain

$$\begin{aligned} &a + y^2 = xy \\ &y^2 - x \cdot y + a = 0 \end{aligned}$$

Applying Vieta's Theorem to the quadratic equation, we have that $y_1 \cdot y_2 = a$ and that

$$x = (y_1 + y_2) \tag{2.67}$$

Second, let us cube both sides:

$$x^3 = \left(\frac{a}{y} + y\right)^3$$

$$x^3 = \frac{a^3}{y^3} + \frac{3a^2}{y} + 3ay + y^3 \tag{2.68}$$

On the other hand, we can obtain

$$-3ax = -3a\left(\frac{a}{y} + y\right) = -\frac{3a^2}{y} - 3ay \tag{2.69}$$

Adding (2.68) and (2.69) we obtain that

$$x^3 - 3ax = \frac{a^3}{y^3} + y^3.$$

Now, the given equation can be written as

$$\frac{a^3}{y^3} + y^3 = 2b \tag{2.70}$$

Equation (2.70) becomes a quadratic equation after substitution:

$$z = y^3 \tag{2.71}$$

$$z^2 - 2bz + a^3 = 0$$

$$z_1 = b + \sqrt{b^2 - a^3} \tag{2.72}$$

$$z_2 = b - \sqrt{b^2 - a^3}$$

If $b^2 > a^3$, then using (2.71) we will find the corresponding y for each z given by (2.72), and then applying (2.67) we will get one unique solution for x as follows:

$$x = \sqrt[3]{b + \sqrt{b^2 - a^3}} + \sqrt[3]{b - \sqrt{b^2 - a^3}} \tag{2.73}$$

Remark Despite the fact that the idea of solving a cubic equation of this type is very similar to the previous problem, there is an obvious difference here that depends on the expression under the square root of formulas (2.72). We will get real values of z if and only if the discriminant of a quadratic equation is nonnegative, $\frac{D}{4} = b^2 - a^3 \geq 0$.

The following cases are valid.

Case 1 $b^2 = a^3$, then we have only one value for z, $z = b$ and one value of y and x, respectively:

$$y = \sqrt[3]{b}$$
$$x = 2\sqrt[3]{b}$$

This means that the polynomial function has only one X-intercept at the indicated x (and two complex conjugate roots).

Case 2

$$b^2 > a^3. \tag{2.74}$$

If (2.74) holds, then from (2.71) the solutions for y will be written as

$$y_1 = \sqrt[3]{b + \sqrt{b^2 - a^3}}$$
$$y_2 = \sqrt[3]{b - \sqrt{b^2 - a^3}}$$

Please also notice that under condition (2.74) both roots for y are positive because $b > \sqrt{b^2 - a^3}$. Next, let us see what solutions can be obtained for x:

It follows from (2.67) that

$$x = y_1 + y_2 > 0,$$

which would lead us to the real solution (2.73).

Moreover, in this case the cubic equation has only one real root and two complex roots.

Case 3 $b^2 < a^3$. In this case the discriminant of a quadratic equation (2.72) is negative, and then $\sqrt{b^2 - a^3} = i \cdot s,\ i = \sqrt{-1}$

$$x = \sqrt[3]{-b + i \cdot s} + \sqrt[3]{-b - i \cdot s}$$

It is not obvious but in this case, the cubic function has three distinct real roots and three X-intercepts. However, the roots cannot be written in radicals (irreducible case).

An example of such a polynomial is $x^3 - 9x - 8 = 0$ $\left(a = 3, b = 4, 4^2 < 3^3\right)$.

Obviously, it has an integer solution $x_1 = -1$ because $(-1)^3 - 9(-1) - 8 = 0$. Two other irrational real roots can be obtained by using synthetic division of the polynomial by $(x + 1)$:

$$x^3 - 9x - 8 = (x + 1)\left(x^2 - x - 8\right) = 0$$

The second factor (quadratic function) gives two additional real roots: $x_{2,3} = \frac{1 \pm \sqrt{33}}{2}$.

Let us see what would happen to our equation if we try to solve it using the substitution (2.66):

$$\begin{aligned}
&x = \frac{3}{y} + y \\
&y^6 - 8y^3 + 27 = 0 \\
&z = y^3 \\
&y = \sqrt[3]{z} \\
&z^2 - 8z + 27 = 0 \\
&z_{1,2} = 4 \pm \sqrt{11} \cdot i \\
&y_1 = \sqrt[3]{4 + \sqrt{11}i};\ y_2 = \sqrt[3]{4 - \sqrt{11}i} \\
&x = \sqrt[3]{4 + \sqrt{11}i} + \sqrt[3]{4 - \sqrt{11}i}
\end{aligned}$$

Of course, it is hard to recognize three real roots $x_1 = -1, x_{2,3} = \frac{1 \pm \sqrt{33}}{2}$ in the answer obtained above. However, if you studied complex numbers, then you probably remember that there are three different cubic roots of a complex number. Thus, the last formula actually represents three real zeroes.

Calculus Question for Case 2 Is there always only one X-intercept for the function $p(x) = x^3 - 3ax - 2b$, $\left(a > 0, b > 0 \text{ and } b^2 > a^3\right)$?

Consider again the first derivative of $p(x)$:

$$p^{'}(x) = 3x^2 - 3a = 3\left(x - \sqrt{a}\right)\left(x + \sqrt{a}\right)$$

Since $a > 0$, the derivative takes zero values at $x = -\sqrt{a}, x = \sqrt{a}$ and the function is decreasing on $x \in (-\sqrt{a}, \sqrt{a})$ and is increasing on $x \in (-\infty, -\sqrt{a}) \cup (\sqrt{a}, \infty)$.

Using an inequality between geometric and arithmetic means applied to (2.66) and remembering that $y > 0$, we obtain that the following is true:

$$x = \frac{a}{y} + y \geq 2\sqrt{\frac{a}{y} \cdot y} = 2\sqrt{a}$$

Therefore, the unique real root satisfying this inequality is positive. The function $p(x) = x^3 - 3ax - 2b$ has a negative Y-intercept and does not have other X-intercepts.

Problem 81 Solve the equation $x^3 - 15x = 144$.

Solution This equation fits the required substitution $x = \frac{5}{y} + y$ for $a = 5$, $b = 72$. It can be rewritten as $y^6 - 144y^3 + 125 = 0$ (biquadratic type equation). Because $72^2 > 5^3$, we could solve the equation or simply use formulas (2.73):

$$y_1 = \sqrt[3]{b + \sqrt{b^2 - a^3}} = \sqrt[3]{72 + \sqrt{5059}} > 0$$
$$y_2 = \sqrt[3]{b - \sqrt{b^2 - a^3}} = \sqrt[3]{72 - \sqrt{5059}} > 0$$

The answer can be written as $x = y_1 + y_2 = \sqrt[3]{72 + \sqrt{5059}} + \sqrt[3]{72 - \sqrt{5059}}$

Answer $x = \sqrt[3]{72 + \sqrt{5059}} + \sqrt[3]{72 - \sqrt{5059}}$.

Problem 82 Consider a polynomial equation $x^3 + 2x^2 + x + a = 0$. Explain how you would solve it. Answer the following questions:

a. What values of a parameter a make this equation solvable using the Rational Zero Theorem? Give an example and solve the equation. Explain how three, two, or one real zeros of the function depend on the value of a.
b. If the Rational Zero Theorem is not applicable, then the equation can be solved by factoring or by other methods presented above. However, those methods require that there be no x^2 term. What substitution can you use in order to rewrite it in the form $z^3 + pz + q = 0$? Let $a = 11$. Solve the equation.

Table 2.2 Problem 82

x	a	$x^3+2x^2+x+a=0$	$(x-a)\cdot p_2(x)$	Real zeros
1	-4	$x^3+2x^2+x-4=0$	$(x-1)(x^2+3x+4)$	$x=1$
-1	0	$x^3+2x^2+x=0$	$x(x+1)^2$	$x_1=0, x_{2,3}=-1$
2	-18	$x^3+2x^2+x-18=0$	$(x-2)(x^2+4x+9)$	$x=2$
-2	2	$x^3+2x^2+x+2=0$	$(x+2)(x^2+1)$	$x=-2$

Solution (a) Consider $x^3+2x^2+x+a=0$. We can rewrite the equation as $a=-(x^3+2x^2+x)=-f(x)$. It is clear that there are infinitely many values of x for which this equation can be solved using the Rational Zero Theorem. For example, Table 2.2 demonstrates some possible pairs of x and a: (1, -4); (-1, 0); (2, -18); (-2, 2). You can continue this table.

Next we will investigate how behavior of the function depends on the value of its parameter a.

Obviously, if $a=0$, then the function $y=x^3+2x^2+x$ has two X-intercepts; at $x=-1$ the function has its local maximum and its graph is tangent to the X-axis.

Can we get an irreducible case with three real roots?

Let us use a calculus approach now and find the first derivative:

$$f(x)=x^3+2x^2+x+a$$
$$f'(x)=3x^2+4x+1$$
$$f'(x)=0$$
$$x=-1, x=-\frac{1}{3}$$

Therefore the function is increasing on $x\in(-\infty,\,-1)\cup\left(-\frac{1}{3},\infty\right)$ and decreasing on $x\in\left(-1,\,-\frac{1}{3}\right)$. Assume that $a>0$. Then it will have one real zero on the interval $-\frac{1}{3}<x<0$.

In order for $f(x)=x^3+2x^2+x+a$ to have two additional real zeros, then one root must be on the interval $x\in\left(-1,\,-\frac{1}{3}\right)$ and the other one is on the interval $x\in(-\infty,\,-1)$. Thus on each interval of monotonic increasing and decreasing, the function must have one zero.

This can be achieved if the following inequality is true:

$$f\left(-\frac{1}{3}\right)\cdot f(-1)<0,$$

which means that the function takes opposite signs at its two critical points.

Figure 2.14a Cubic function with three X-intercepts

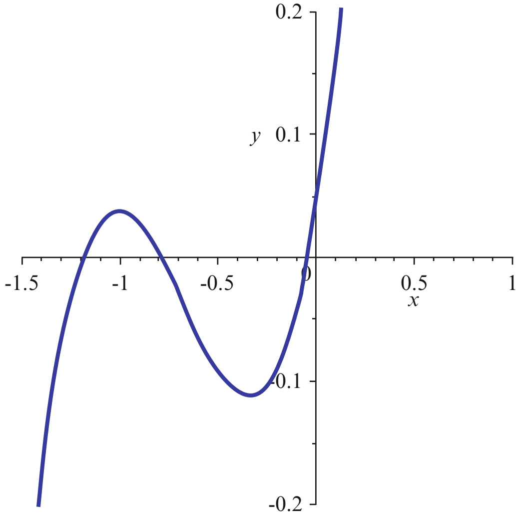
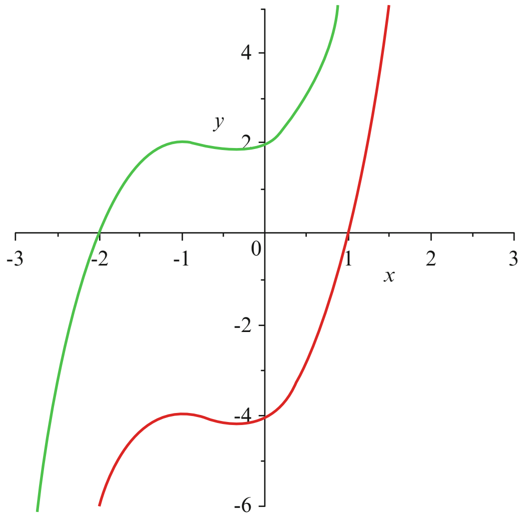


Figure 2.14b One X-intercept (Problem 82)

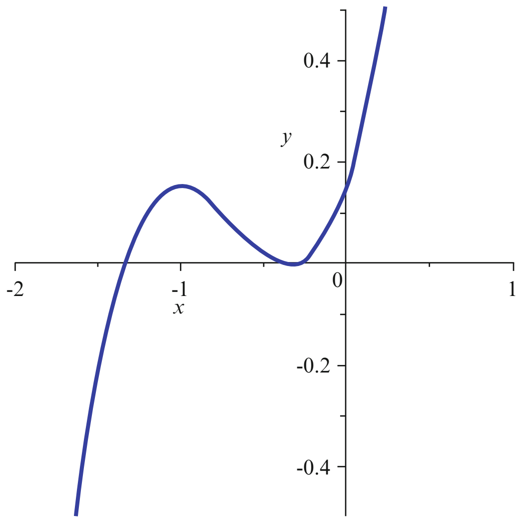


Replacing $x = -\dfrac{1}{3}$ and $x = -1$ we obtain the inequality for a parameter a:

$$a \cdot \left(a - \frac{4}{27}\right) < 0$$
$$0 < a < \frac{4}{27}$$

If a satisfies the inequality, then the function $f(x) = x^3 + 2x^2 + x + a$ will have precisely three real zeroes. For example, we can choose $a = \dfrac{1}{27}$ and state that the

Figure 2.14c Graph of $f(x) = x^3 + 2x^2 + x + \dfrac{4}{27}$ (two X-intercepts)



equation $x^3 + 2x^2 + x + \frac{1}{27} = 0$ or the equivalent equation $27x^3 + 54x^2 + 27x + 1 = 0$ has three real zeroes. Below you can see the graph (Figure 2.14a) of the function $f(x) = x^3 + 2x^2 + x + \frac{1}{27}$. We cannot find its zeros using the ideas described above because we need to eliminate a quadratic term first. Detailed methods of finding its zeroes analytically and the methods of solving a general type cubic equation are presented in the following section.

If $f(-1) \cdot f\left(-\frac{1}{3}\right) > 0$ (or $a \cdot \left(a - \frac{4}{27}\right) > 0$) then the quantities keep the same sign, either both positive or both negative. This will happen if either $a < 0$ or $a > \dfrac{4}{27}$.

If $f(-1) \cdot f\left(-\dfrac{1}{3}\right) = 0$ (or $a \cdot \left(a - \dfrac{4}{27}\right) = 0$), then one of the two possible X-intercepts of the function is also either its local maximum ($x = -1,\ a = 0$) or local minimum $\left(x = -\dfrac{1}{3}\ a = \dfrac{4}{27}\right)$.

If $a < 0$, then the function $f(x) = x^3 + 2x^2 + x + a$ will have only one positive real zero, to the right of its local minimum $x = -\dfrac{1}{3}$ and if $a = 0$ then $f(x)$ has two real zeros, $x = 0$ and $x = -1$ (of multiplicity two). If $a > \dfrac{4}{27}$, the function will have one negative real zero, to the left of the local maximum, $x = -1$, and if $a = \dfrac{4}{27}$, the function will have two real zeros, $x = -\dfrac{4}{3}$ and $x = -\dfrac{1}{3}$ (of multiplicity two).

Two cases with one X-intercept for $a = -4 < 0, x = 1$ and for $a = 2 > \dfrac{4}{27}$, $x = -2$ are shown in Figure 2.14b as red and green curves, respectively.

There are only two values of the parameter a $\left(a = 0, a = \dfrac{4}{27}\right)$ at which our function will have two X-intercepts. Let us confirm our statement for $a = \dfrac{4}{27}$.

The first case ($a = 0$) is trivial and can be seen in Table 2.2.

At $a = \dfrac{4}{27}$, $f(x) = x^3 + 2x^2 + x + \dfrac{4}{27}$ and its zeros can be found by solving the following equation:

$$27x^3 + 54x^2 + 27x + 4 = 0.$$

Making the substitution $z = 3x$ we obtain a simpler equation in z:

$$z^3 + 6z^2 + 9z + 4 = 0.$$

Using the Rational Zero Theorem we can find that $z = -1$ $\left(x = -\frac{1}{3}\right)$ is the root. After synthetic division we obtain that the modified equation can be factored as $(z + 1)(z + 1)(z + 4) = 0$. Hence zero $x = -\frac{1}{3}$ has multiplicity two (coincides with the local minimum of the function) and another real zero for $f(x)$ is $x = -\frac{4}{3} < -1$ (to the left of the local maximum). See Figure 2.14c.

(b) One of this type of problem with $a = 11$ and hence the equation $x^3 + 2x^2 + x + 11 = 0$ can be rewritten in the form $z^3 - 3z + 295 = 0$ by first introducing $x = y - \frac{a}{3}$ and then by replacing $z = 3y$. Since $a = 11 > \frac{4}{27}$, we can state that the equation will have only one real zero, which is located to the left of $x = -1$. This equation is solved later as Problem 84.

2.6.4 Cardano's Formula for Cubic Equations

Consider a polynomial equation of the third order:

$$x^3 + ax^2 + bx + c = 0 \tag{2.75}$$

Our goal is to solve this equation if for example the Rational Zero Theorem or other methods cannot be applied to find its zeroes.

First, we will learn what substitution would *eliminate the quadratic term* making the equation look like one that we solved in the previous sections $(y^3 + py + q = 0)$. Next, we will derive Cardano's formula. **Cardano** in 1545 published methods developed by **Tartaglia** for solving a cubic equation of a general type.

Let

$$x = y - \frac{a}{3} \tag{2.76}$$

Then

$$\left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c = 0$$

We will show all work here:

$$\begin{aligned}
&y^3 - 3y^2 \cdot \frac{a}{3} + 3y \cdot \frac{a^2}{9} - \frac{a^3}{27} + a\left(y^2 - \frac{2ya}{3} + \frac{a^2}{9}\right) + by - \frac{ab}{3} + c = 0 \\
&y^3 - \frac{a^2 y}{3} + \frac{2a^3}{27} + by - \frac{ab}{3} + c = 0 \\
&y^3 + \left(b - \frac{a^2}{3}\right) \cdot y + \left(c + \frac{2a^3}{27} - \frac{ab}{3}\right) = 0
\end{aligned}$$

Let $p = b - \frac{a^2}{3}$, $q = c + \frac{2a^3}{27} - \frac{ab}{3}$; then we can rewrite the previous equation in the form

$$y^3 + py + q = 0, \tag{2.77}$$

where $p, q \in R$ (real numbers).

After the familiar substitution

$$y = z - \frac{p}{3z} \tag{2.78}$$

we obtain the following

$$y^3 = \left(z - \frac{p}{3z}\right)^3 = z^3 - 3z^2 \cdot \frac{p}{3z} + 3z \cdot \frac{p^2}{9z^2} - \frac{p^3}{27z^3} = z^3 - pz + \frac{p^2}{3z} - \frac{p^3}{27z^3}$$

Now, in terms of the new variable, z, the equation (2.77) has the following form:

$$z^3 - \frac{p^3}{27z^3} + q = 0.$$

This can be changed to a quadratic type polynomial equation in t by letting $t = z^3$, $t^2 = z^6$ as

$$z^6 + q \cdot z^3 - \frac{p^3}{27} = 0 \tag{2.79}$$

From (2.79) we obtain the solutions as

$$z^3{}_{1,2} = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2} = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

After taking a cubic root the solutions look as

$$z_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \quad z_2 = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \tag{2.80}$$

It follows from (2.78) that there are two solutions for y:

$$y_i = z_i - \frac{p}{3z_i}, i = 1, 2.$$

However, as we have already seen, the two answers are the same, and the solution of (2.78) is

$$y = y_1 = y_2 = z_1 + z_2 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \tag{2.81}$$

Formula (2.81) is known as ***Cardano's formula*** for a cubic equation of type (2.77). The value of y depends on the relationships between parameters q and p and their values. This dependence will be discussed later in this section. The corresponding value of x for the general type cubic equation (2.75) can be easily obtained from formula (2.76).

Historical Solution of Cardano-Tartaglia: Vieta's Theorem Approach

In order to solve cubic equation (2.77), we can also use Vieta's Theorem.

Let

$$y = u + v \tag{2.82}$$

$$(u + v)^3 = u^3 + 3u^2v + 3v^2u + v^3 = u^3 + v^3 + 3uv(u + v)$$

Then we can substitute this into (2.77):

$$u^3 + v^3 + 3uv(u + v) + p(u + v) + q = 0 \tag{2.83}$$

Next, we can impose a new condition:

$$p = -3uv \tag{2.84}$$

This would make (2.83) have the following form:

$$u^3 + v^3 = -q \tag{2.85}$$

Equation (2.84) can also be written as

$$u^3 v^3 = -\frac{p^3}{27} \tag{2.86}$$

If we look at (2.84) and (2.85) together, we can recognize the *Reversed Vieta's Theorem* written for the roots

$$t_1 = u^3, t_2 = v^3 \tag{2.87}$$

of a quadratic equation

$$t^2 + q \cdot t - \frac{p^3}{27} = 0, \tag{2.88}$$

such as

$$\begin{cases} t_1 + t_2 = -q \\ t_1 \cdot t_2 = -\dfrac{p^3}{27} \end{cases}$$

The roots of a quadratic equation (2.88) can be found using the discriminant divided by 4 version of the quadratic formula:

$$t_{1,2} = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

I hope that you understand the great "plan" of Cardano-Tartaglia.

1. Using substitution (2.76), you rewrite $x^3 + ax^2 + bx + c = 0$ in terms of a new variable y without quadratic term as $y^3 + py + q = 0$.
2. Write corresponding to it a quadratic equation $t^2 + q \cdot t - \frac{p^3}{27} = 0$ and find its solutions.
3. Because $y = u + v$ and $u^3 = -\dfrac{q}{2} + \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}$ and $v^3 = -\dfrac{q}{2} - \sqrt{\dfrac{q^2}{4} + \dfrac{p^3}{27}}$,

we will have to evaluate the corresponding values of u and v. Of course, this would lead us to Cardano's formula. However, we need to understand that further calculation of the roots will depend strongly on the value of the ***discriminant*** $\frac{D}{4}=\frac{q^2}{4}+\frac{p^3}{27}=\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3$ of the quadratic equation and on the values of parameters q and p. Thus the following cases are possible:

Case 1 (Trivial) $p=q=0$, then $y=0$, then $x=-b$ is the real root of multiplicity 3.

Case 2 If $p=0, q\neq 0$, then $u=0, v=-\sqrt[3]{q}, y=-\sqrt[3]{q}$.

Case 3 If $\frac{D}{4}=\frac{q^2}{4}+\frac{p^3}{27}=0$ $(p<0, q\neq 0)$, then $u=v=-\frac{q}{2}, y=-q$. In this case, we will have either one real root of multiplicity 3 or one real root of multiplicity 2 and one single real root. Hence, the cubic function has one X-intercept or two X-intercepts. Therefore, the function will not change its sign at its intercept, or at one of its intercepts, respectively, but will be tangent to the X-axis at that point.

Algebraically we can show that discriminant of a cubic equation $x^3+px+q=0$ can be written as $D=(x_1-x_2)^2(x_1-x_3)^2(x_3-x_2)^2$. If $D=0$ or if $4p^3+27q^2=0$, then at least two roots of the cubic equation must be the same.

Case 4 If $\frac{D}{4}=\frac{q^2}{4}+\frac{p^3}{27}>0, \quad \Rightarrow \quad y=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$.

In this case, a polynomial of third degree has only one real root and two complex roots.

Case 5 If $\frac{D}{4}=\frac{q^2}{4}+\frac{p^3}{27}<0$ (discriminant is strictly negative), then there are three real roots. This case is called *Casus irreducibilis* and it means that the roots cannot be written in the radical form.

Case 5 needs clarification. Because each complex root has three different cubic roots, in order to find all solutions of the corresponding cubic equation, it is convenient to use Cardano's formula in the form

$$y=z-\frac{p}{3z}=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}-\frac{p}{3\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}} \tag{2.89}$$

Remark Let us now see how well you learned the material of Chapter 1. Please answer the following question: How many X-intercepts does the function $y=f(x)=x^3+x+1$ have?

Of course, you can easily calculate the discriminant: $\frac{D}{4} = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 = \frac{1}{4} + \frac{1}{27} > 0$, and then state that the function has only one X-intercept.

However, we can look at the function as the sum of two increasing functions, $h(x) = x^3, g(x) = x + 1$, and therefore, their sum is also an increasing function with only one real zero.

Problem 83 Solve the equation $27x^3 + 54x^2 + 27x + 1 = 0$.

Solution Denote $t = 3x$, and then $x = \frac{t}{3}$, $t^3 = 27x^3$, $t^2 = 9x^2$ and the equation has the form $t^3 + 6t^2 + 9t + 1 = 0$.

Our next step is to eliminate the quadratic term by following (2.76):

$$t = y - \frac{6}{3} = y - 2$$
$$(y-2)^3 + 6(y-2)^2 + 9(y-2) + 1 = 0$$

After simplification we will obtain the equation in the form $y^3 + py + q = 0$: $y^3 - 3y - 1 = 0$ where $p = -3, q = -1$.

In order to write solutions to this equation, first we will evaluate the discriminant:

$$\frac{D}{4} = \frac{q^2}{4} + \frac{p^3}{27} = \frac{1}{4} + \frac{(-3)^3}{27} = -\frac{3}{4} < 0$$
$$\sqrt{-\frac{3}{4}} = \frac{\sqrt{3}}{2} \cdot i$$

In this case, we have three real irrational roots. Thus, we need to evaluate three different cubic roots of a complex number $\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Using Euler's formula $e^{i\varphi} = \cos\varphi + i\sin\varphi$, we will denote the complex number under the cubic root by z_0 and rewrite it in the exponential form

$$z_0 = \frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{\frac{\pi}{3}\cdot i} = e^{i\cdot\left(\frac{\pi}{3} + 2\pi n\right)}$$

Note, that sine and cosine are periodic functions, and this is why we added the period in the power. Assume that z is the 3rd root of z_0. Then the following is true:

$$\begin{aligned}
&z^3 = z_0\\
&e^{i3\varphi} = e^{i\left(\frac{\pi}{3}+2\pi n\right)}\\
&\varphi_n = \frac{\pi}{9} + \frac{2\pi}{3}\cdot n, \quad n = 0, 1, 2\\
&\varphi_1 = \frac{\pi}{9} \quad \Rightarrow \qquad z_1 = e^{i\,\frac{\pi}{9}}\\
&\varphi_2 = \frac{7\pi}{9} \quad \Rightarrow \qquad z_2 = e^{i\,\frac{7\pi}{9}}\\
&\varphi_1 = \frac{13\pi}{9} \quad \Rightarrow \qquad z_3 = e^{i\,\frac{13\pi}{9}}
\end{aligned}$$

For each z we will find the corresponding value of y (2.89) and then t and x:

$$y_1 = z - \frac{p}{3z} = e^{i\frac{\pi}{9}} + \frac{1}{e^{i\frac{\pi}{9}}} = e^{i\frac{\pi}{9}} + e^{-i\frac{\pi}{9}} = 2\cos\frac{\pi}{9}$$

$$x_1 = \frac{y-2}{3} = \frac{2\cos\frac{\pi}{9} - 2}{3} = \frac{2}{3}\left(\cos\frac{\pi}{9} - 1\right) \approx -0.04$$

$$y_2 = z_2 - \frac{p}{3z_2} = e^{i\frac{7\pi}{9}} + \frac{1}{e^{i\frac{7\pi}{9}}} = e^{i\frac{7\pi}{9}} + e^{-i\frac{7\pi}{9}} = 2\cos\frac{7\pi}{9}$$

$$x_2 = \frac{y_2-2}{3} = \frac{2\cos\frac{7\pi}{9} - 2}{3} = \frac{2}{3}\cdot\left(\cos\frac{7\pi}{9} - 1\right) \approx -1.173$$

$$y_3 = z_3 - \frac{p}{3z_3} = e^{i\frac{13\pi}{9}} + \frac{1}{e^{i\frac{13\pi}{9}}} = e^{i\frac{13\pi}{9}} + e^{-i\frac{13\pi}{9}} = 2\cos\frac{13\pi}{9} = 2\cos\frac{5\pi}{9}$$

$$x_3 = \frac{y_3-2}{3} = \frac{2}{3}\left(\cos\frac{5\pi}{9} - 1\right) \approx -.782$$

Answer $x_1 = \frac{2}{3}\left(\cos\frac{\pi}{9} - 1\right), \quad x_2 = \frac{2}{3}\left(\cos\frac{5\pi}{9} - 1\right), \quad t_3 = \frac{2}{3}\left(\cos\frac{7\pi}{9} - 1\right).$

Note that *the trigonometric form of the three real zeroes* is more appropriate than a complex radical form (see Figure 2.15). This way we can always see our zeroes exactly and even check that the answer is true on a graphing calculator. Maple, for example, for this equation, gives the following roots presented in Figure 2.16.

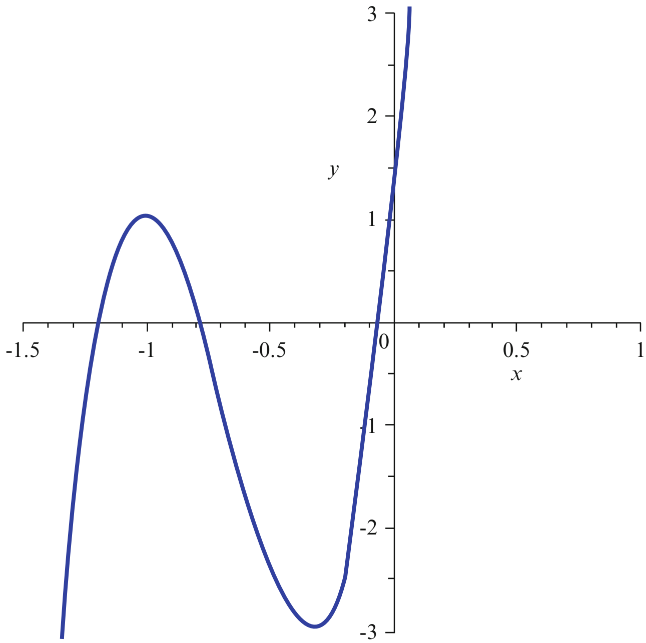


Figure 2.15 The graph of $f(x) = 27x^3 + 54x^2 + 27x + 1$

$$
\begin{aligned}
&\frac{1}{6}\left(4+4\mathrm{I}\sqrt{3}\right)^{1/3}+\frac{2}{3\left(4+4\mathrm{I}\sqrt{3}\right)^{1/3}}-\frac{2}{3},\ -\frac{1}{12}\,(4\\
&\quad+4\mathrm{I}\sqrt{3})^{1/3}-\frac{1}{3\left(4+4\mathrm{I}\sqrt{3}\right)^{1/3}}-\frac{2}{3}\\
&\quad+\frac{1}{6}\,\mathrm{I}\sqrt{3}\left(\frac{1}{2}\left(4+4\mathrm{I}\sqrt{3}\right)^{1/3}-\frac{2}{\left(4+4\mathrm{I}\sqrt{3}\right)^{1/3}}\right),\\
&-\frac{1}{12}\left(4+4\mathrm{I}\sqrt{3}\right)^{1/3}-\frac{1}{3\left(4+4\mathrm{I}\sqrt{3}\right)^{1/3}}-\frac{2}{3}\\
&\quad-\frac{1}{6}\,\mathrm{I}\sqrt{3}\left(\frac{1}{2}\left(4+4\mathrm{I}\sqrt{3}\right)^{1/3}-\frac{2}{\left(4+4\mathrm{I}\sqrt{3}\right)^{1/3}}\right)
\end{aligned}
$$

Figure 2.16 MAPLE 15 answer for $27x^3 + 54x^2 + 27x + 1 = 0$

More about the trigonometric approach to solving cubic equations and *Casus irreducibilis* can be found in Chapter 3 of this book.

Problem 84 Find solutions to the equation $x^3 + 2x^2 + x + 11 = 0$.

Solution Based on the Rational Zero Theorem, this equation does not have integer zeroes. Let us use Cardano's formula:

$$y = x + b = x + \frac{2}{3}$$

$$y^3 - \frac{1}{3}y + \frac{295}{27} = 0$$

$$27y^3 - 9y + 295 = 0$$

Let $z = 3y$ and we obtain a cubic equation without a quadratic term:

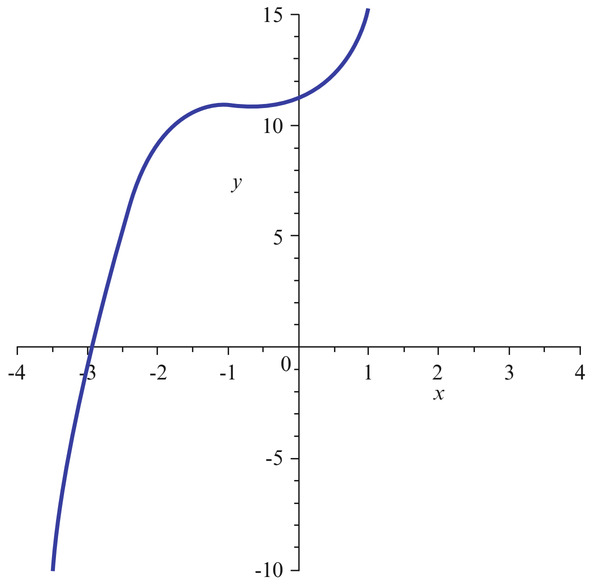
$$z^3 - 3z + 295 = 0$$

Define the corresponding $p = -3$ and $q = 295$. Because $\frac{D}{4} = \frac{295^2}{4} - 1 > 0$, we have only one real root and one X-intercept (see Figure 2.17).

In order to make simplifications, let us use a difference of squares formula (2.10) again. Thus $\sqrt{\frac{295^2}{4} - 1} = \sqrt{\frac{295^2 - 2^2}{2^2}} = \sqrt{\frac{293 \cdot 297}{2^2}} = \sqrt{\frac{293 \cdot 9 \cdot 33}{2^2}} = \frac{3\sqrt{9669}}{2}$

$$z = \sqrt[3]{-\frac{295}{2} + \frac{3\sqrt{9669}}{2}} + \sqrt[3]{-\frac{295}{2} - \frac{3\sqrt{9669}}{2}} < -\sqrt[3]{295} \approx -6.656 \left(\sqrt{\frac{295^2}{4} - 1} < \frac{295}{2} \right)$$

Figure 2.17 The graph of $y = x^3 + 2x^2 + x + 11$



The answer can be simplified as

$$x = \frac{z-2}{3} = \frac{\sqrt[3]{\frac{3\sqrt{9669}-295}{2}} - \sqrt[3]{\frac{3\sqrt{9669}+295}{2}} - 2}{3} \approx -2.88.$$

Problem 85 Solve the equation $x^3 - 12x^2 + 21x - 11 = 0$.

Solution If this equation has any rational zeroes, then they must be integers and factors of (-11). If we check for $x = 1, -1, 11$, and -11, then we can state that this equation does not have integer roots. Let us use Cardano's formula and rewrite this equation in terms of y:

$$y^3 + py + q = 0$$
$$y = x - 4$$

Let us substitute $x = y + 4$ into the original equation:

$$(y+4)^3 - 12(y+4)^2 + 21(y+4) - 11 = 0$$
$$y^3 - 27y - 55 = 0$$
$$p = -27, q = -55$$

Calculate the discriminant $\frac{D}{4} = \frac{q^2}{4} + \frac{p^3}{27} = \frac{(-55)^2}{4} + \frac{(-27)^3}{27} = \frac{55^2}{4} - \frac{4 \cdot 27^2}{4} = \frac{(55-54)(55+54)}{4} = \frac{109}{4} > 0$.

From this, we know that this equation will have only one real root:

$$y = \sqrt[3]{\frac{55}{2} + \frac{\sqrt{109}}{2}} + \sqrt[3]{\frac{55}{2} - \frac{\sqrt{109}}{2}} = \frac{\sqrt[3]{55+\sqrt{109}} + \sqrt[3]{55-\sqrt{109}}}{\sqrt[3]{2}}$$

$$x = y + 4 = \frac{\sqrt[3]{55+\sqrt{109}} + \sqrt[3]{55-\sqrt{109}}}{\sqrt[3]{2}} + 4 \approx 6.012$$

Answer $x = \frac{\sqrt[3]{55+\sqrt{109}} + \sqrt[3]{55-\sqrt{109}}}{\sqrt[3]{2}} + 4$.

In present times, because of the existence of calculators and computers, Cardano's formula is rarely used in the classroom. However, in my opinion, the

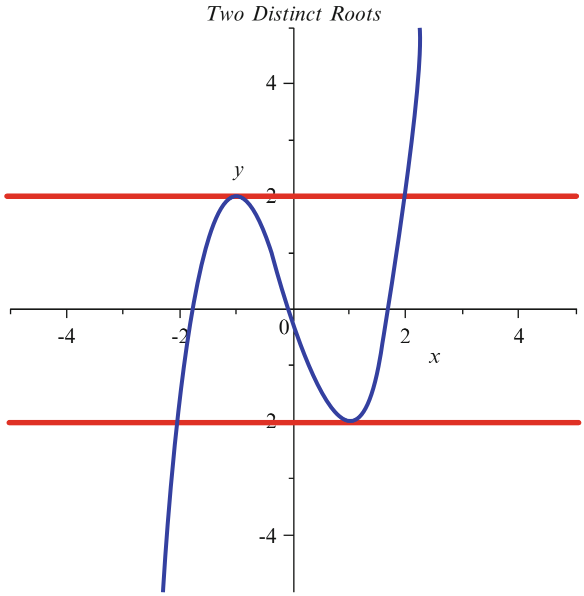


Figure 2.18 Two roots of the equation $x^3 - 3x = a$

beauty of how the formula was derived can be a great learning tool for future mathematicians, even in this computer generation.

Problem 86 For what value of a does the equation $x^3 - 3x = a$ have two real, distinct solutions?

Solution Method 1: Let us sketch the graph of the cubic function $y = f(x) = x^3 - 3x$. This function has a local maximum at $x = -1$ and a local minimum at $x = 1$. The line $y = g(x) = a$ has two common points with the graph of the cubic function at $a = 2$ or $a = -2$ (only at the points tangent to the extrema) (Figure 2.18).

Method 2: Consider a cubic equation $x^3 - 3x - a = 0$. It is in the reduced form $(x^3 + px + q = 0,\ \ p = -3,\ \ q = -a)$, so we can apply Cardano's formula. First, we will evaluate the discriminant and set it equal to zero:

$$\frac{D}{4} = \frac{q^2}{4} + \frac{p^3}{27} = \frac{a^2}{4} - 1 = \frac{(a-2)(a+2)}{4} = 0$$

Because the discriminant is zero at $a = 2,\ \ a = -2$, at these values both $y_1 = x^3 - 3x + 2$ shown in blue and $y_2 = x^3 - 3x - 2$ shown in green have precisely two X-intercepts (Figure 2.19).

Answer $a = 2$, $a = -2$.

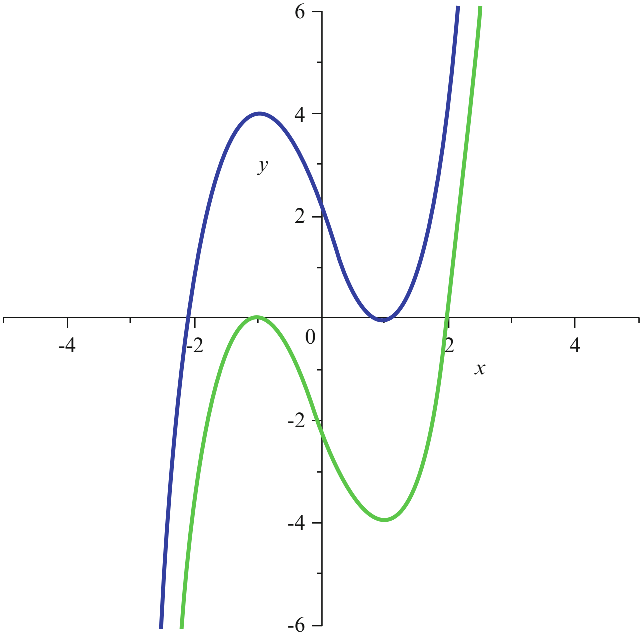


Figure 2.19 Two cubic curves

2.7 Higher Order Equations: Methods of Ferrari and Euler

So far we have learned how to solve quadratic equations, cubic equations, and equations of higher degree with symmetric coefficients. We also know that any quadratic equation can be solved any time for any coefficients using, for example, the quadratic formula. What about a quartic equation of general type? Can we find a formula to express the roots in terms of the equation's coefficients? In the 16th century a solution to a general type polynomial equation of fourth degree was obtained by the Italian mathematician **Ferrari**, who was a student of Cardano. Let us briefly describe his method.

Assume that we need to solve the equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + c = 0$$

Using a substitution

$$x = y - \frac{b}{a},$$

we will obtain a new equation without a cubic term:

$$y^4 + 2py^2 + 2qy + r = 0 \tag{2.90}$$

If in the equation above, the coefficient q is zero; then we can solve it by introducing a new variable $z = y^2$ and then solve it as a quadratic equation in z.

However, usually $q \neq 0$, and then we need to find a different idea.

Let us separate the terms of (2.90) as follows:

$$y^4 + 2py^2 = -2py - r$$

And complete the square on the left-hand side:

$$\left(y^2 + p\right)^2 = p^2 - 2qy - r$$

The idea is to introduce a new variable, t, so we can also complete the square on the right-hand side:

$$\left(y^2 + p + t\right)^2 = p^2 - 2qy - r + 2pt + 2ty^2 + t^2 \tag{2.91}$$

Consider $p^2 - 2qy - r + 2pt + 2ty^2 + t^2$ as a quadratic function in variable y:

$$f(y) = 2ty^2 - 2q \cdot y + \left(t^2 + 2pt + p^2 - r\right) \tag{2.92}$$

If the discriminant (or $\frac{D}{4}$ for an even coefficient of y) of $f(y)$ equals zero, then (2.92) will be rewritten as $2t \cdot (y - y^*)^2$, where $y^* = \dfrac{q}{2t}$ and (2.91) will take an appropriate form $(y^2 + p + t)^2 = 2t\left(y - \dfrac{q}{2t}\right)^2$, which can be written as a difference of squares:

$$\left(y^2 + p + t\right)^2 - \left(\sqrt{2t}\left(y - \frac{q}{2t}\right)\right)^2 = 0$$

And then factored and decomposed into two quadratic equations!

$$\left(y^2 + p + t + \sqrt{2t} \cdot \left(y - \frac{q}{2t}\right)\right) \cdot \left(y^2 + p + t - \sqrt{2t} \cdot \left(y - \frac{q}{2t}\right)\right) = 0$$

$$\left(y^2 + \sqrt{2t} \cdot y + p + t - \frac{q}{\sqrt{2t}}\right) \cdot \left(y^2 + \sqrt{2t} \cdot y - p + t + \frac{q}{\sqrt{2t}}\right) = 0 \tag{2.93}$$

Now, we need to find the unknown function, t. Let us solve the equation $\dfrac{D}{4} = 0$:

$$\frac{D}{4} = q^2 - 2t\left(t^2 + 2pt + p^2 - r\right) = 0 \tag{2.94}$$

The equation (2.94) is called the ***resolvent***. Note that (2.94) is a cubic equation in variable t that can also be rewritten as

$$2t^3 + 4pt^2 + \left(p^2 - r\right)t - q^2 = 0$$

Solving this equation, we will find the unknown t, substitute it in (2.93), and find y.

Unfortunately, all attempts of mathematicians to find a method of solving polynomial equations of general type of degree higher than four had failed. Finally, in the early 1800s, Norwegian mathematician **Abel** (1802–1829) proved that expressing the solution of a general equation of degree five or higher by any formula is impossible.

Let us practice Ferrari's method in solving the following problem.

Problem 87 Solve the equation $y^4 + 4y - 1 = 0$.

Solution Comparing the equation with (2.90), we will find the corresponding coefficients as

$$p = 0,\ q = 3,\ r = -1.$$

Introduce a new variable t and complete the squares as follows:

$$\left(y^2 + t\right)^2 = -4y + 1 + t^2 + 2ty^2$$

$$2ty^2 - 4y + t^2 + 1 = 2t(y - y^*)^2$$

$$y^* = \frac{1}{t}$$

$$\frac{D}{4} = 4 - 2t\left(t^2 + 1\right) = 0$$

Thus the resolvent is a cubic equation:

$$t^3 + t - 2 = 0$$

From the Rational Zero Theorem we can find the required root,

$$t = 1 \Rightarrow y^* = 1.$$

Next, our equation can be rewritten as

$$\left(y^2 + 1\right)^2 = \left(\sqrt{2} \cdot (y - 1)\right)^2$$

And then decomposed into two quadratic equations that can be solved separately:

$$y^2 + \sqrt{2}y + \left(1 - \sqrt{2}\right) = 0$$

$$y_{1,2} = \frac{-\sqrt{2} \pm \sqrt{4\sqrt{2} - 2}}{2} = \frac{-1 \pm \sqrt{\sqrt{8} - 1}}{\sqrt{2}}$$

and

$$y^2 - \sqrt{2}y + 1 + \sqrt{2} = 0$$

$$y_{3,4} = \frac{1 \pm i\sqrt{\sqrt{8} + 1}}{\sqrt{2}}$$

Since this equation does not give real roots, the 4th degree polynomial equation has only two real zeroes.

Answer

$$y_{1,2} = \frac{-1 \pm \sqrt{\sqrt{8} - 1}}{\sqrt{2}}.$$

Remark With some experience, this equation can be written as

$$\left(y^2 + 1\right)^2 = \left(\sqrt{2} \cdot (y - 1)\right)^2$$

If we do the following steps

$$\begin{aligned} y^4 &= 1 - 4y \\ y^4 + 2y^2 + 1 &= 2y^2 + 1 + 1 - 4y \\ \left(y^2 + 1\right)^2 &= 2(y^2 - 2y + 1) \\ \left(y^2 + 1\right)^2 &= \left(\sqrt{2}(y - 1)\right)^2 \end{aligned}$$

then the solution can be found much more easily and without any resolvent.

Problem 88 Solve the equation $\left((x+2)^2 + x^2\right)^3 = 8x^4 \cdot (x+2)^2$.

Solution This equation is a polynomial of 6th degree. Let us first simplify it by introducing a new variable:

$$y = x + 1 \tag{2.95}$$

After substitution and simplification, we will obtain

$$\left((y+1)^2 + (y-1)^2\right)^3 = 8(y-1)^4 \cdot (y+1)^2$$

This can be further simplified by applying the difference of squares formula to the right-hand side and by using the binomial square formulas:

$$\begin{aligned}&(2y^2+2)^3 = 8(y-1)^2[(y-1)(y+1)]^2\\&(y^2+1)^3 = (y-1)^2 \cdot (y^2-1)^2\end{aligned}$$

The last equation can be written as

$$2y^5 + 4y^4 - 4y^3 + 4y^2 + 2y = 0$$

and can be factored as

$$\begin{aligned}&y \cdot (y^4 + 2y^3 - 2y^2 + 2y + 1) = 0\\&1.\ \ y = 0\ \ (x = -1) \qquad 2.\ y^4 + 2y^3 - 2y^2 + 2y + 1 = 0\end{aligned}$$

Next, we will focus on solving a quartic equation. We probably could use Ferrari's method here and this problem is left as HW Problem 11 for you. However, I want you to notice that we obtained a *symmetric* polynomial of even degree, so we can get solutions faster by dividing the equation by y^2:

$$y^2 + 2y - 2 + \frac{2}{y} + \frac{1}{y^2} = 0 \tag{2.96}$$

and then by introducing a new variable as was done in the earlier section:

$$z = y + \frac{1}{y} \tag{2.97}$$

$$z^2 = y^2 + 2 + \frac{1}{y^2} \tag{2.98}$$

Substituting (2.97) and (2.98) into (2.96) we obtain a quadratic equation in z:

$$z^2 + 2z - 4 = 0$$
$$z_{1,2} = -1 \pm \sqrt{5}$$

Now for each value of z, we need to solve (2.97) and find the corresponding y:

1. $y + \frac{1}{y} = -1 + \sqrt{5}$

$$y^2 - \left(\sqrt{5} - 1\right)y + 1 = 0$$

$$y_{1,2} = \frac{\sqrt{5} - 1 \pm \sqrt{2 - 2\sqrt{5}}}{2}$$

2. $y + \frac{1}{y} = -1 - \sqrt{5}$

$$y^2 + \left(\sqrt{5} + 1\right)y + 1 = 0$$

$$y_{3,4} = \frac{-\sqrt{5} - 1 \pm \sqrt{2 + 2\sqrt{5}}}{2}$$

Only the last two answers are real solutions. If we substitute those values of y in (2.95), we will obtain the corresponding values of x.

Answer $x_1 = -1; x_{2,3} = \frac{-\sqrt{5}-3}{2} \pm \sqrt{\frac{\sqrt{5}+1}{2}}$.

It is interesting that the original polynomial looked like one of 6th degree, but the 6th degree terms were cancelled and we obtained three real roots and two complex roots for a polynomial function of the 5th degree by factoring and solving a symmetric quartic equation.

2.7.1 Euler's Method for Solving a Quartic Equation

Consider the equation $y^4 + py^2 + qy + r = 0$.

Then find its cubic resolvent as

$$z^3 + 2pz^2 + \left(p^2 - 4r\right)z - q^2 = 0$$

and its roots z_1, z_2, z_3.

Then solutions to a quartic equation are

$$\begin{aligned} y_1 &= \frac{1}{2}(\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}) \\ y_2 &= \frac{1}{2}(\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3}) \\ y_3 &= \frac{1}{2}(-\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}) \\ y_4 &= \frac{1}{2}(-\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3}) \end{aligned} \tag{2.99}$$

The roots can be complex. Note that the signs of the roots are selected by the rule

$$\sqrt{z_1} \cdot \sqrt{z_2} \cdot \sqrt{z_3} = -q \tag{2.100}$$

Problem 89 Solve the equation $x^4 - 8x^3 + 18x^2 - 27 = 0$ using Euler's method.

Solution Let $x = y + 2$. Next, we have

$$y^4 - 6y^2 + 8y - 3 = 0 \tag{2.101}$$

Its cubic resolvent is

$$\begin{aligned} &z^3 - 12z^2 + 48z - 64 = 0 \\ &(z - 4)^3 = 0 \end{aligned} \tag{2.102}$$

Then the roots of (2.102) are $z_1 = z_2 = z_3 = 4$. Considering (2.100) and that $q = 8$, we obtain
$\sqrt{z_1} \cdot \sqrt{z_2} \cdot \sqrt{z_3} = -8$, and then $\sqrt{z_1} = \sqrt{z_2} = \sqrt{z_3} = -2$.
Therefore, applying (2.99), the roots of (2.101) are

$$y_1 = -3, y_{2,3,4} = 1.$$

Answer $x_1 = -1, x_{2,3,4} = 3$.

In general, any polynomial function of the fourth degree with four real roots and unit leading coefficient can be factored as $p(x) = x^4 + ax^3 + bx^2 + cx + d$ $= (x - x_1)\cdot\, (x - x_2) \cdot (x - x_3) \cdot (x - x_4) = \prod_{i=1}^{4} (x - x_i)$

Then any two factors can be coupled to obtain a product of two quadratic functions.

For example, $x^4 - 5x^2 + 4 = (x - 1)(x + 1)(x - 2)(x + 2)$

$$= [(x - 1)(x + 1)] \cdot [(x - 2)(x + 2)] = (x^2 - 1)(x^2 - 4)$$
$$= [(x - 1)(x - 2)] \cdot [(x + 1)(x + 2)] = (x^2 - 3x + 2)(x^2 + 3x + 2)$$

In general, we can solve the following problem.

Problem 90 A polynomial of the fourth degree with four real roots x_1, x_2 and x_3, x_4 can be factored as $p(x) = x^4 + ax^3 + bx^2 + cx + d = (x^2 + px + q)(x^2 + rx + s)$. Evaluate $A = \left(x_1^2 + 1\right)\left(x_2^2 + 1\right)\left(x_3^2 + 1\right)\left(x_4^2 + 1\right)$ in terms of p, q, r, s.

Solution Without loss of generality, we can assume that x_1, x_2 and x_3, x_4 are the roots of the first and the second quadratic equations, respectively. Now applying Vieta formulas for the 1$^{\text{st}}$ quadratic, we have

$$\begin{aligned}\left(x_1^2 + 1\right)\left(x_2^2 + 1\right) &= x_1^2 x_2^2 + x_1^2 + x_2^2 + 1 = (x_1 x_2 - 1)^2 + (x_1 + x_2)^2\\ &= (q-1)^2 + p^2.\end{aligned}$$

Similarly for the 2nd quadratic equation we obtain

$$\left(x_3^2 + 1\right)\left(x_4^2 + 1\right) = (s-1)^2 + r^2.$$

So the product of all four quantities is

$$\left(x_1^2 + 1\right)\left(x_2^2 + 1\right)\left(x_3^2 + 1\right)\left(x_4^2 + 1\right) = \left((q-1)^2 + p^2\right)\left((s-1)^2 + r^2\right).$$

Answer $A = \left((q-1)^2 + p^2\right)\left((s-1)^2 + r^2\right)$.

2.8 Miscellaneous Problems on Polynomials

Problem 91 Find a polynomial of minimal degree having a maximum value of 6 at $x = 1$ and minimum value of 2 at $x = 3$.

Solution Given a maximum value of 6 at $x = 1$ and a minimum value of 2 at $x = 3$, then the polynomial must be of third degree and its derivative must be a quadratic function that can be written as $f'(x) = a(x-1)(x-3)$ for some real number a. The following is true:

$$f'(x) = a\left(x^2 - 4x + 3\right) = ax^2 - 4ax + 3a$$

Then $f(x) = \dfrac{ax^3}{3} - \dfrac{4ax^2}{2} + 3ax + c$.

Evaluating $f(1)$ and $f(3)$, we obtain

$$f(1) = \frac{a}{3} - \frac{4a}{2} + 3a + c = 6 \text{ and } f(3) = \frac{27a}{3} - \frac{36a}{2} + 9a + c = 2$$

$$f(1) = \frac{4}{3}a + c = 6 \text{ and } f(3) = c = 2$$

Now substitute $c = 2$ into $f(1) = \frac{4}{3}a + c = 6$ and find $a = 3$.

Therefore, the polynomial of minimal degree is $f(x) = x^3 - 6x^2 + 9x + 2$.

Problem 92 Prove that there is no polynomial $p(x)$ with integer coefficients such that $p(7) = 5$ and $p(15) = 9$.

Solution We will prove this by contradiction.

Assume that such a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$ does exist, and then $P(7) = a_n \cdot 7^n + a_{n-1} \cdot 7^{n-1} + \ldots + a_0$ and $P(15) = a_n \cdot 15^n + a_{n-1} \cdot 15^{n-1} + \ldots + a_0$.

Subtracting the two polynomials, we obtain $P(15) - P(7) = a_n \cdot (15^n - 7^n) + a_{n-1} \cdot \left(15^{n-1} - 7^{n-1}\right) + \ldots + a_1(15 - 7) = 9 - 5 = 4$.

Because $\forall k,\ 15^k - 7^k$ is divisible by $15 - 7 = 8$ (recall Corollary 1 at the beginning of this chapter), it follows that $P(15) - P(7) = 4$ itself is divisible by 8, which is not true.

We obtained a contradiction. Hence, such a polynomial does not exist.

Problem 93 Let $p(x)$ be a polynomial with integer coefficients taking value 5 at five integer values of x. Prove that $p(x)$ does not have integer zeroes.

Solution Proof by contradiction: Assume that such a polynomial $P(x) = Q(x)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5) + 5$ exists and that here x_i are five distinct integers and $Q(x)$ is a polynomial with integer coefficients of lesser degree than $P(x)$. Assume that $x = n$ is an integer zero of the polynomial $P(x)$. If $P(n) = 0$, then $-5 = Q(n)(n - x_1)(n - x_2)(n - x_3)(n - x_4)(n - x_5)$. Therefore $Q(n)$ is also an integer.

Note that each of the factors on the right-hand side must be a factor of 5 and x_i are five distinct integers. However, 5 is a prime number and only has four integer factors $\{\pm 1; \pm 5\}$, a contradiction. Therefore $P(x)$ does not have integer zeroes.

Problem 94 Prove the equation $x^5 + ax^4 + bx^3 + c = 0$, where $a, b, c \in R$, $c \neq 0$ has at least two complex, not real zeroes.

Solution This problem requires knowledge of calculus and derivatives. Let this equation have only real roots and denote them $a_1, a_2, \ldots, a_s$ $(a_1 < a_2 < \ldots < a_s)$. Because some of the roots can have multiplicity more than one, we will also assume that $k_1, k_2, \ldots, k_s$ are the corresponding multiplicities of the roots and then the following must be satisfied: $k_1 + k_2 + \ldots + k_s = 5$.

For example, some polynomial function of fifth degree with only real zeroes can have the form

$$h(x) = (x - a_1)^2(x - a_2)^3; g(x) = (x - a_1)^3(x - a_2)(x - a_3);$$
$$p(x) = (x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5); s(x) = (x - a_1)^4 \cdot (x - a_2)$$

Denote $f(x) = x^5 + ax^4 + bx^3 + c$. If $k_i > 1$, then a_i is also a zero of the first derivative of the function, $f'(x)$, of multiplicity $(k_i - 1)$. Thus the sum of the multiplicities of the zeroes of $f'(x)$ among the roots $a_1, a_2, \ldots, a_s$ equals $(5 - s)$.

Moreover, $f'(x)$ must have zero b_i between a_i and a_{i+1} (in total it will have $\geq s - 1$ such zeroes), and if at least one other root, b_i, has multiplicity more than one, then the sum of the multiplicities of the roots for the first derivative of the function will be $\geq (5 - s) + s = 5$, which is impossible because all zeroes of $f'(x)$ with multiplicity more than one are among the zeroes $(a_1, \ldots, a_s)$ of function $f(x)$. On the other hand, 0 is a zero of the first derivative with multiplicity two and 0 is not the root of $f(x)$. Thus we obtained a contradiction and based on the Fundamental Theorem of Algebra, the given function will have at least two complex conjugate roots, and maximum three real zeroes.

Remark The reason why we considered zeroes of the function with multiplicities more than one is that if we assume that the function has five distinct zeroes, we would obtain a contradiction right away. If

$$\begin{aligned} f(x) = x^5 + ax^4 + bx^3 + c \Rightarrow f'(x) &= 5x^4 + 4ax^3 + 3bx^2 \\ &= x^2(5x^2 + 4ax + 3b). \end{aligned}$$

Depending on the discriminant of the quadratic function $\left(\frac{D}{4} = 4a^2 - 15b\right)$, the first derivative will have either one zero $(x = 0)$ of multiplicity two, two zeroes (both with multiplicity two), or three zeroes (one (0) with multiplicity two, two other zeroes with multiplicity one). Consider the last case when the derivative can be written as $f'(x) = x^2 \cdot (x - b_1)(x - b_2)$. By Rolle's Theorem, if a function $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), and if $f(a) = f(b)$, then there must be at least one zero of the first derivative, $c \in (a, b)$. Because $x = 0$ is not a zero of $f(x)$, the function can have at most three zeroes, $a_1 < a_2 < a_3$, such that $b_1 \in (a_1, a_2), b_2 \in (a_2, a_3)$.

Problem 95 Given $c_0 + \frac{c_1}{2} + \frac{c_2}{3} + \ldots + \frac{c_n}{n+1} = 0$. Prove that the polynomial $p(x) = c_n x^n + \ldots + c_2 x^2 + c_1 x + c_0$ has at least one real zero.

Solution Let $f(x) = c_0 x + \frac{c_1}{2}x^2 + \frac{c_3}{3}x^3 + \ldots + \frac{c_n}{n+1}x^{n+1}$; then its derivative is $f'(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n = p(x)$.

Evaluate $f(0) = 0$ and $f(1) = 0$. Because $f(x)$ is continuous on $[0, 1]$ and $f(0) = f(1) = 0$, by Rolle's Theorem $\exists\ \xi \in (0, 1)$ such that $f'(\xi) = 0$. Therefore, the polynomial $p(x)$ has at least one real zero.

Problem 96
How many polynomials are there of the form
$p(x) = x^3 + ax^2 + bx + c$ such that their roots are a, b, and c?

Solution Let $p(x) = x^3 + ax^2 + bx + c$ with roots $x = a, x = b, x = c$.

Applying Vieta's Theorem for a cubic equation, we obtain the system

$$\begin{cases} a + b + c = -a \\ bc + ac + ab = b \\ abc = -c \end{cases}$$

Solving for a, b, c, all possible solutions are

$$\begin{cases} c(ab + 1) = 0 \\ 2a + b + c = 0 \\ bc + ac + ab = b \end{cases}$$

From the first equation of the system, the following cases are possible:

1. $c = 0$.

$$\begin{cases} c = 0 \\ 2a + b = 0 \\ b(a - 1) = 0 \end{cases} \Leftrightarrow \left[\begin{array}{l} \{c = 0, b = 0, a = 0 \\ \begin{cases} c = 0 \\ a = 1 \\ b = -2 \end{cases} \end{array} \right.$$

2. $ab = -1$.

$$\begin{cases} a = -\dfrac{1}{b} \\ -\dfrac{2}{b} + b + c = 0 \\ bc - \dfrac{c}{b} - 1 = b \end{cases} \Rightarrow \begin{cases} c\left(b - \dfrac{1}{b}\right) = b + 1 \\ c = \dfrac{2}{b} - b \\ a = -\dfrac{1}{b} \end{cases} \Leftrightarrow \begin{cases} c\left(b^2 - 1\right) = (b+1)b \\ c = \dfrac{2}{b} - b \\ a = -\dfrac{1}{b} \end{cases} \Leftrightarrow$$

$$\begin{cases} (b+1) \cdot (c(b-1) - b) = 0 \\ c = \dfrac{2}{b} - b \\ a = -\dfrac{1}{b} \end{cases}$$

Using the first equation of the last system, we obtain two cases:

$$\begin{cases} b = -1 \\ a = 1 \\ c = -1 \end{cases}$$

or

$$\begin{cases} c(b-1) - b = 0 \\ c = \dfrac{2}{b} - b \\ a = -\dfrac{1}{b} \end{cases} \Leftrightarrow \begin{cases} \left(\dfrac{2}{b} - b\right)(b-1) - b = 0 \\ c = \dfrac{2}{b} - b \\ a = -\dfrac{1}{b} \end{cases} \Leftrightarrow \begin{cases} b^3 - 2b + 2 = 0 \\ c = \dfrac{2}{b} - b \\ a = -\dfrac{1}{b} \end{cases}$$

Let us investigate the number of real roots of the first, cubic equation in this last system and evaluate its discriminant:

$$\frac{D}{4} = \frac{q^2}{4} + \frac{p^3}{27} = \frac{19}{27} > 0.$$

Because the discriminant is positive, the cubic equation has only one real, negative root. It can be shown that $b_0 < -\sqrt{\dfrac{2}{3}}$. Therefore there are four such polynomials with the following coefficients:

Answer $(a, b, c) : \{(0,0,0), (1, -2, 0), (1, -1, -1), \left(-\dfrac{1}{b_0}, b_0, \dfrac{2}{b_0} - b_0\right)$.

2.9 Homework on Chapter 2

1. Solve $2014x^2 - 2013x - 1 = 0$.
   Solution: Do not use the quadratic formula, the numbers would be too big. Instead, find that $x = 1$ is one root, and then by Vieta's Theorem the second root is 1/2014.
2. Solve $37x^2 + 73x - 2 = 0$.

   Hint: Use Vieta's Theorem to obtain $x = -2$, $x = 1/37$.

3. Solve $2x^3 + x - 18 = 0$.

   Solution: The cubic function on the left is monotonically increasing, so it will have only one real zero. By checking we see that $x = 2$ is zero. Therefore, this is the only zero of the function.

4. Solve $116x^2 + 115x = 1$.

   Hint: We can mentally find one of the roots. It is $x = -1$, and then the second roots is found by Vieta's Theorem as $x = -1/116$.

5. For a polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$ with four real zeroes, x_1, x_2, x_3, x_4, the following relationship is valid: $b - d \geq 5$. Find the minimal value of the product $A = \left(x_1^2 + 1\right)\left(x_2^2 + 1\right)\left(x_3^2 + 1\right)\left(x_4^2 + 1\right) \to \min$.

   Solution: (One of the methods). As we did in Problem 91, the polynomial can be written as the product of two quadratic functions:

$$p(x) = x^4 + ax^3 + bx^2 + cx + d = \left(x^2 + px + q\right)\left(x^2 + rx + s\right)$$

   And the value of A in terms of p, q, r, s was obtained earlier as

$$A = \left((q-1)^2 + p^2\right)\left((s-1)^2 + r^2\right) \tag{2.103}$$

   Comparing coefficients of the original polynomial and its factorized form, we have that

$$b = q + s + pr, \quad d = qs$$

   Therefore, the given inequality is

$$\begin{aligned}
&b - d \geq 5 \\
&q + s + pr - qs \geq 5 \\
&pr \geq qs - q - s + 1 + 4
\end{aligned}$$

The last inequality can be written as

$$pr \geq (q-1)\cdot(s-1)+4$$

or as

$$pr-(q-1)(s-1)\geq 4 \tag{2.104}$$

Next, we will prove the following statement:

$$\left(x^2+y^2\right)\left(z^2+t^2\right)\geq (yt-xz)^2,\quad x,y,z,t\in R. \tag{2.105}$$

Note that the validity of a similar inequality $(x^2+y^2)(z^2+t^2)\geq (yt+xz)^2$ follows from the Cauchy- Bunyakovsky inequality (see Chapter 4).
Proof: Expanding both sides we have

$$\begin{aligned}&x^2z^2+x^2t^2+y^2z^2+y^2t^2\geq y^2t^2-2xyzt+x^2z^2\\&x^2t^2+y^2z^2\geq -2xyzt\\&(xt+yz)^2\geq 0,\end{aligned}$$

which is true.

Therefore, the inequality (2.105) is true.

Using (2.105) and (2.104), we can rewrite (2.103) as follows:

$$A=\left((q-1)^2+p^2\right)\left((s-1)^2+r^2\right)\geq (pr-(q-1)(s-1))^2\geq 4^2=16.$$

Therefore the smallest value of A is 16. In order to verify this case we can consider $p(x)=(x-1)^4=x^4-4x^3+6x^2-4x+1$.

6. Prove that if $x^2+ax+b=0$ has rational root, then it is an integer.

   Hint: The proof is similar to the proof of Theorem 18. Assume that $x=\frac{p}{q},\ (p,q)=1$, substitute it into the quadratic equation, isolate the quadratic term on the left, then multiply both sides of the equation by q, and demonstrate that the equation will not have solutions in integers.
7. Solve the equation $(x^2-x+8)(x^2-6-x)=120$.

   Hint: Look at the substitution $y=x^2-x+8\ \Rightarrow\ x^2-x-6=y-14$ and the equation become quadratic in y:

$$\begin{aligned}&y(y-14)=120\\&y^2-14y-120=0,\ \text{etc.}\end{aligned}$$

   Answer: 4; -3.

8. Find all X-intercepts of the function $f(x) = x^{2015} + 35x - 36$.

   Hint: Since $f(x)$ consists of two monotonically increasing functions, it takes each value once, including the zero value. We can see that $f(1) = 0$.

   Answer: $x = 1$.

9. Solve the equation $(x + 1)(x + 2)(x + 3)(x + 4) = 840$.

   Hint: Combine the two middle factors and the two outer factors separately and then introduce a new variable:

$$
\begin{aligned}
&(x+1)(x+2)(x+3)(x+4) = 840\\
&(x^2+5x+4)(x^2+5x+6) = 840\\
&y = x^2+5x+4\\
&y \cdot (y+2) = 840
\end{aligned}
$$

   Answer: $x = -8$; 3.

10. Solve the equation $(x + 4)(x + 6)(x + 8)(x + 10) = 5760$.

    Answer: -16; 2.

11. Solve the equation $y^4 + 2y^3 - 2y^2 + 2y + 1 = 0$ using Ferrari's method.

    Answer: $y_{1,2} = \dfrac{-\sqrt{5} - 1 \pm \sqrt{2 + 2\sqrt{5}}}{2}$.

12. Solve the equation $(x^2 + 3x + 2)(x^2 - 20 + 3x) = 240$.

    Answer: -7; 4.

13. Solve the equation: $x^4 - 5x^3 + 6x^2 - 5x + 1 = 0$.

    Answer: $2 \pm \sqrt{3}$.

14. Solve the equation $x^4 + 3x^3 - 8x^2 + 3x + 1 = 0$

    Answer: 1; $-2.5 \pm \sqrt{5.25}$.

15. Solve the equation $4x^4 - 8x^3 - 37x^2 - 8x + 4 = 0$.

    Hint: This is a symmetric equation.

    Answer: -2; $-1/2$; $\dfrac{9 \pm \sqrt{65}}{4}$.

16. One of the roots of $4x^4 - 12x^3 + 13x^2 - 12 + a = 0$ equals 2. Find the value of a and other roots of the equation.

    Answer: $a = 2, x_1 = 2, x_2 = \dfrac{1}{2}$.

17. Prove that if p is the root of the equation $ax^4 + bx^3 + cx^2 + bx + a = 0$, $a \neq 0$ then $\dfrac{1}{p}$ is also the root of it.

Proof: Because the equation is a symmetric quartic equation, we can find its solutions by introducing a new variable, $y = x + \frac{1}{x}$. Without solving this equation, we notice that if we substitute the solution, $x = p$, then we will obtain $y(p) = p + \frac{1}{p} = y\left(\frac{1}{p}\right)$. Therefore $x = \frac{1}{p}$ is also a solution.

18. Solve the equation $(x^2 - x - 1)^2 - x^3 = 5$.

Solution: Instead of raising this quantity to the second power, we will factor it as follows, first by writing 5 as $4 + 1$:

$$(x^2 - x - 1)^2 - 4 = x^3 + 1$$
$$(x^2 - x - 1)^2 - 2^2 = (x + 1)(x^2 - x + 1)$$

Applying a difference of squares formula and then factoring out the common factor we obtain

$$\left(x^2 - x - 1 - 2\right)\left(x^2 - x - 1 + 2\right) = (x + 1)\left(x^2 - x + 1\right)$$
$$\left(x^2 - x + 1\right)\left(x^2 - 2x - 4\right) = 0$$
$$x_{1,2} = 1 \pm \sqrt{5}$$

19. Solve the equation $(x^2 + x)^2 + \left|x^2 + x\right| - 2 = 0$.

Solution: We will use the following important substitution:

$$|z|^2 = z^2, z = \left|x^2 + x\right|$$
$$z^2 + z - 2 = 0$$
$$z = 1, \ \ z = -2$$
$$\left|x^2 + x\right| = 1$$
$$x^2 + x - 1 = 0$$
$$x_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$$

20. Find a polynomial function that has the given zeroes and Y-intercept. Zeroes: 1, -1, i; Y-intercept (0, -5).

Answer: $y = 5x^4 - 5$.

21. List all possible rational zeroes of the function. Sketch the graph and explain it using the leading coefficient test: $y = f(x) = 4x^3 + x^2 - x - 4$.

Answer: Possible rational zeros: $\pm\frac{1}{4}, \pm\frac{1}{2}, \pm 1, \pm 2, \pm 4$.
Notice that $f(1) = 0$ and then $x = 1$ is a zero and using synthetic division we can factor the function as $(x - 1)(4x^2 + 5x + 4) = 0$. The function has only one

X-intercept. The Y-intercept is -4. Since the leading coefficient is positive and the degree is odd, the function rises to the right and falls to the left.

22. Peter expanded a binomial $(x+y)^n$ and lost his notes. He remembers that the second term was 240, the third was 720, and the fourth was 1080. Please help him to find x, y, and n.

Hint: Using (2.30) we can write the following system for the second, third, and fourth terms:

$$\begin{cases} nx^{n-1}y = 240 \\ \dfrac{n(n-1)}{2}\cdot x^{n-2}y^2 = 720 \\ \dfrac{n(n-1)(n-2)}{6}\cdot x^{n-3}y^3 = 1080 \end{cases}$$

Solving the system we obtain the answer.
Answer: $x=2$, $y=3$, $n=5$.

23. Evaluate $S_1 = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots$ and $S_2 = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$.

Hint: Adding the two sums, reorganizing terms and using (2.31) we obtain $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots + \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots = 2^n$. Half of the sum is the answer.

Solution: Earlier we proved Theorem 21 that can be written as

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} = \binom{n}{0}x^n + \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} + \binom{n}{3}x^{n-3} + \ldots$$

Using it we can state that the following is also true:

$$\begin{aligned}(x-1)^n &= \sum_{k=0}^{n} \binom{n}{k}(-1)^k x^{n-k} \\ &= \binom{n}{0}x^n - \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2} - \binom{n}{3}x^{n-3} + \ldots\end{aligned}$$

Substituting $x=1$ in the formula above, we see that the left-hand side becomes zero and on the right-hand side, the binomial coefficients with even lower index will have the "+" sign and all binomial coefficients with odd lower index will have the "−" sign. Moving all negative terms to the left side, we obtain that the sum of all even binomial coefficients, S_1, equals the sum of all odd binomial coefficients, S_2. Because by (2.31) the total sum of all binomial coefficients equals 2^n, each sum (S_1 or S_2) is half of it or 2^{n-1}.

Answer: $S_1 = S_2 = 2^{n-1}$.

24. Prove that any even polynomial function with positive coefficients is concave up (convex) and has only one extreme point.

    Proof: Because the given even polynomial function with positive coefficients can be written as $f(x) = \sum_{n=0} a_{2n}x^{2n}$, its derivative $g(x) = f'(x) = 2\sum_{n=1} na_{2n}x^{2n-1}$ is an odd function, such that $g(-x) = -g(x)$. Because each coefficient is positive, the derivative function is strictly increasing and has only one zero, $x = 0$. At this zero the first derivative changes its sign from minus to plus, then at $x = 0$, the given polynomial function has a local minimum. We can confirm that the function is concave up because the second derivative of $f(x)$ is always positive. Therefore, $f(x)$ has only one extreme point, the minimum at $x = 0$.

25. Solve the equation $x^4 - 8x^2 - 4x + 3 = 0$.

    Hint: Apply the Rational Zero Theorem and find two integer solutions $x = -1$ and $x = 3$. Use synthetic division to find two other irrational roots.
    Answer: $-1; 3; -1 \pm \sqrt{2}$.

26. Find all X-intercepts of the function $f(x) = x^4 + x^3 - 5x^2 + 2$.

    Hint: Because, by the Rational Zero Theorem, any integer factors of 2 are not the solutions of the equation, apply Ferrari's method.

    Answer: $x_{1,2} = -1 \pm \sqrt{3};\ \ x_{3,4} = \dfrac{1 \pm \sqrt{5}}{2}$.

27. Find the X-intercept of the cubic function $y = x^3 + 9x - 2$.

    Solution: Because the equation $x^3 + 9x - 2 = 0$ fits a special type of cubic equation $x^3 + 3ax = 2b$, $a = 3$, $b = 1$ we will denote $x = \frac{3}{y} - y$ and make a substitution. The equation will be rewritten as a quadratic in $z = y^3$ as $z^2 + 2z - 27 = 0$, which has the following solutions:

$$z_1 = -1 + \sqrt{28} \Rightarrow y_1 = \sqrt[3]{-1 + \sqrt{28}}$$
$$z_2 = -1 - \sqrt{28} \Rightarrow y_2 = -\sqrt[3]{1 + \sqrt{28}}$$

    Next, we will find $x = -(y_1 + y_2) = \sqrt[3]{1 + \sqrt{28}} - \sqrt[3]{\sqrt{28} - 1}$.

    Answer: $x = \sqrt[3]{1 + \sqrt{28}} - \sqrt[3]{\sqrt{28} - 1}$.

28. Find all points of the intersection of cubic function $f(x) = 8x^3 - 3x - 4$ with parabola $g(x) = 3x^2 - 3$.

    Solution: If the two functions intersect, then the following is true:

$$8x^3 - 3x - 4 = 3x^2 - 3$$
$$8x^3 - 3x^2 - 3x - 1 = 0$$

Of course, we can solve this equation, but instead, let us rewrite the last equation as $8x^3 + x^3 = x^3 + 3x^2 + 3x + 1$ in order to complete the cube on the right-hand side:

$$9x^3 = (x+1)^3$$
$$\left(\sqrt[3]{9}x\right)^3 - (x+1)^3 = 0$$
$$x = \frac{1}{1-\sqrt[3]{9}}$$

Answer: $x = \dfrac{1}{1-\sqrt[3]{9}}$.

29. For what values of a one of the roots of $x^2 - \frac{15}{4} \cdot x + a^3 = 0$ is the square of the other root?

    Answer: $a = -\dfrac{5}{2}; \dfrac{3}{2}$.

30. Prove that $x^2 - 1995x + 10a + 1 = 0$ cannot have integer roots for any integer parameter $a \in Z$.

    Proof: Using Vieta's Theorem we have the system

$$\begin{cases} x_1 \cdot x_2 = 10a + 1 \\ x_1 + x_2 = 1995 \end{cases}$$

    The sum of the roots is odd (1995), then one the roots must be odd, and the other root must be even number. This contradicts the first equation of the system, because the product of odd and even number must be an even, but $10a + 1$ is odd!

31. Given a function $f(x) = ax^2 + bx + c$, such that $f(1) < 0, f(2) > 3, f(3) < 6$. Find the sign of the leading coefficient, a.

    Solution: Using the conditions of the problem we have the system

$$\begin{cases} a + b + c < 0 \\ 4a + 2b + c > 3 \\ 9a + 3b + c < 6 \end{cases}$$

    Adding the first and the last inequalities we obtain

$$10a + 4b + 2c < 6$$
$$a + (4a + 2b + c) < 3$$

    Because the quantity inside parentheses is greater than 3, we can state that a is negative.

    Answer: $a < 0$.

32. Solve the equation $x^4 - 4x^3 + 7x^2 - 2x - 5 = 0$.

Solution: Let $x = y + 1$ and substitute it into the equation $y^4 + y^2 + 4y - 3 = 0$.

This equation has a cubic resolvent as $z^3 + 2z^2 + 13z - 16 = 0$, which by the Rational Zero Theorem has zero $z = 1$ (please check by substitution). Two other roots will be obtained after applying synthetic division of the resolvent by $(z - 1)$. However, one can factor the quartic equation in y as follows:

$$(y^2 - y + 3)(y^2 + y - 1) = 0$$
$$y^2 - y + 3 = 0 \quad y^2 + y - 1 = 0$$
$$y_{1.2} = \frac{1 \pm i\sqrt{11}}{2} \quad y_{3.4} = \frac{-1 \pm \sqrt{5}}{2}$$
$$x_{1,2} \notin R \qquad x_{3,4} = \frac{1 \pm \sqrt{5}}{2}$$

Answer: Two real solutions are $x = \dfrac{1 \pm \sqrt{5}}{2}$.

33. Solve the equation $x^4 - 8x^3 + 18x^2 - 27 = 0$.

Answer: $x_1 = -1, x_{2,3,4} = 3$.

Solution: Method 1: Using a standard method for quartic equations, let $x = y + 2$; then for the equation $y^4 - 6y^2 + 8y - 3 = 0$ we obtain the cubic resolvent as $z^3 - 12z^2 + 48z - 64 = 0$.

We can recognize the cube of the difference here as

$(z - 4)^3 = 0$, then $z = 4$, etc.

Method 2: Applying the Rational Zero Theorem we notice that $x = 1$ is the solution; then using synthetic division we factor the original polynomial equation as $(x + 1)(x - 3)(x - 3)(x - 3) = 0$.

34. Find real solutions of the equation $2\left(x^2 + \dfrac{1}{x^2}\right) - 7\left(x + \dfrac{1}{x}\right) = 0$.

Hint: Make a substitution $y = x + \dfrac{1}{x}$, then $y^2 - 2 = x^2 + \dfrac{1}{x^2}$, and the equation becomes a quadratic $2y^2 - 7y - 4 = 0$ in variable y, etc.

Answer: $x = 2 \pm \sqrt{3}$.

35. Solve a problem proposed by Cardano: $13x^2 = x^4 + 2x^3 + 2x + 1$.

Hint: Rewrite the equation as $x^4 + 2x^3 - 13x^2 + 2x + 1 = 0$, recognize a recurrent (symmetric) equation of even degree, then divide both side by x^2, and substitute $y = x + \dfrac{1}{x}$.

Solution: Applying the Rational Zero Theorem we conclude that there are no rational zeroes. However, the equation is symmetric and after an appropriate substitution we obtain

$$y^2 + 2y - 15 = 0$$
$$(y + 5)(y - 3) = 0$$
$$y = -5, y = 3.$$

Next, we will find the corresponding values of x:

1.
$$x + \frac{1}{x} = -5$$
$$x^2 + 5x + 1 = 0$$
$$x = \frac{-5 \pm \sqrt{21}}{2}$$

2.
$$x + \frac{1}{x} = 3$$
$$x^2 - 3x + 1 = 0$$
$$x = \frac{3 \pm \sqrt{5}}{2}$$

It is interesting that all roots are real and irrational (two negative and two positive), so they could not be obtained by the Rational Zero Theorem. Introduction of a new variable allows to decouple the equation into two quadratic equations $(x^2 + 5x + 1)(x^2 - 3x + 1) = 0$.

Answer: There are four real irrational roots: $x_{1,2} = \frac{-5 \pm \sqrt{21}}{2}$, $x_{3,4} = \frac{3 \pm \sqrt{5}}{2}$.

36. Numbers a, b, c are three of the four real zeroes of the function $y = x^4 - ax^3 - bx + c$. Find all such polynomial functions.

    Hint: See Problem 96.

    Answer: $x^4 - ax^3$; $x^4 - ax^3 - x + a$; $x^4 - x^3 + x - 1$; $x^4 + x$

37. Find the relationship between the coefficients of the polynomial $ax^3 + bx^2 + cx + d = 0$, if it is known that the sum of two of its roots equals the product of these roots.

    Hint: Use Vieta's Theorem for cubic equation (2.47).

    Solution: Let x_1, x_2, x_3 be zeros of the given cubic equation. Denote $z = x_1 + x_2 = x_1 \cdot x_2$. Then Vieta's Theorem can be written as

$$\begin{cases} z + x_3 = -\dfrac{b}{a} \\ z + x_3 \cdot z = -\dfrac{d}{a} \\ z \cdot x_3 = \dfrac{c}{a} \end{cases}$$

Dividing the second and third equations of the system we obtain $\dfrac{1+x_3}{x_3} = -\dfrac{c}{d}$ that can be solved for x_3 as

$$x_3 = -\frac{d}{d+c}. \tag{2.106}$$

From the first equation of the system we have

$$z = -\frac{b}{a} - x_3$$
$$z = -\frac{b}{a} + \frac{d}{d+c}$$

or

$$x_1 \cdot x_2 = \frac{-bd - bc + ad}{a(d+c)} \tag{2.107}$$

Multiplying (2.106) and (2.107), using the last equation of the system and after simplification, we obtain the requested formula:

$$\begin{aligned} x_1x_2x_3 &= -\frac{d}{d+c} \cdot \frac{-bd-bc+ad}{a(d+c)} = -\frac{d}{a} \\ (d+c)^2 &= ad - bc - bd \\ d^2 + 2dc + c^2 + bc + bd &= ad \\ (c+d)(b+c+d) &= ad. \end{aligned}$$

Answer: $(c+d)(b+c+d) = ad$.

38. Prove that if $x^2 + ax + b = 0$ has a rational root, then it is an integer.

Hint: The proof is similar to the proof of Theorem 18.

Proof: Assume that the root is rational, such as $x = \dfrac{p}{q}$, $(p, q) = 1$. Substituting it into the equation, we obtain the following:

$$\left(\frac{p}{q}\right)^2 + a \cdot \frac{p}{q} + b = 0.$$

Multiplying both sides by q and separating the terms, we have

$$\frac{p^2}{q} = -ap - bq$$

It follows from this equation that the right-hand side is an integer while the left-hand side is a fraction that cannot be reduced (because p and q are relatively prime). We have obtained a contradiction. Therefore, the given equation cannot have rational roots, only integer roots.

39. Prove that the positive root of the equation $x^5 + x = 10$ cannot be a rational number.

    Hint: Use Theorem 18.

    Proof: Using Theorem 18, we know that if the given equation has a rational root, then it must be an integer. You can use the knowledge of Chapter 1 material and the fact that the function $f(x) = x^5 + x - 10$ is increasing over the entire domain and then it can have one real zero. Next, using the corollary from the Intermediate Value Theorem, we notice that $g(x) = x^5 + x > 10$, if $x = 1.6$ and that $g(x) = x^5 + x < 10$, if $x = 1.5$. There is not an integer number between 1.5 and 1.6. Therefore, this equation does not have any rational roots.

40. Prove that the equation $x^3 + ax^2 - b = 0$, $b > 0$ can have only one positive root.

    Proof: Assume a contradiction and that the equation has two positive roots, for example, $x_2 > 0$, $x_3 > 0$. Then obviously $x_2x_3 > 0$, $x_2 + x_3 > 0$. Applying Vieta's Theorem, we have the following system:

$$\begin{cases} x_1x_2 + x_1x_3 + x_2x_3 = 0 \\ x_1x_2x_3 = b > 0 \end{cases}$$

From the first equation we obtain that

$$x_1(x_2 + x_3) = -x_2x_3$$
$$x_1 = -\frac{x_2x_3}{x_2 + x_3} < 0$$

If $x_1 < 0$, then it will make the product of three roots negative, $x_1x_2x_3 < 0$, which contradicts the second equation of the system. Therefore, if $b > 0$, then the given equation $x^3 + ax^2 - b = 0$ can have only one positive root.

41. Solve the equation $x^3 + x^2 + x = -\frac{1}{3}$.

Solution: If we multiply both sides by 3 we obtain

$$3x^3 + 3x^2 + 3x + 1 = 0.$$

The Rational Zero Theorem would not help to find any zeros. However, after rewriting the first term as $2x^3 + x^3$, the equation can be factored as a sum of two cubes:

$$\begin{aligned}
&\left(\sqrt[3]{2}x\right)^3 + (x^3 + 3x^2 + 3x + 1) = 0\\
&\left(\sqrt[3]{2}x\right)^3 + (x + 1)^3 = 0\\
&\left(\sqrt[3]{2}x + x + 1\right)\left(\left(\sqrt[3]{2}x\right)^2 - \sqrt[3]{2}x(x + 1) + (x + 1)^2\right) = 0\\
&\qquad x \cdot \left(\sqrt[3]{2} + 1\right) + 1 = 0\\
&\qquad x = -\frac{1}{1 + \sqrt[3]{2}}
\end{aligned}$$

Answer: $x = -\dfrac{1}{1 + \sqrt[3]{2}}$.

42. Ann noticed that the numbers $11^2 = 121$ and $11^3 = 1331$ look like the rows of the Pascal's triangle; then she found an explanation to this fact and quickly predicted the value of 11^4. Can you do the same? Explain, please.

Solution: Because $11 = 10 + 1$, then

$$\begin{aligned}
11^2 &= (10 + 1)^2 = C_2^0 10^2 + C_2^1 \cdot 10^1 + C_2^2 \cdot 10^0 = 1 \cdot 10^2 + 2 \cdot 10 + 1 = 121\\
11^3 &= (10 + 1)^3 = C_3^0 10^3 + C_3^1 \cdot 10^2 + C_3^2 \cdot 10 + C_3^3 \cdot 10^0 = 1 \cdot 10^3 + 3 \cdot 10^2\\
&\quad + 3 \cdot 10 + 1 = 1331\\
11^4 &= (10 + 1)^4 = 14641.
\end{aligned}$$

Answer: 14641.

Chapter 3
Problems from Trigonometry

Trigonometry is not a topic that is well understood by many students. In fact, trigonometry is not covered at all in some high schools' curriculum. I notice that many of my students remember the "cool look" of some trigonometric functions (sine wave, etc.) and even the fact of their periodicity but still cannot solve a simple trigonometric equation if its solution requires something more than just knowledge of the main trigonometric identities. For example, when solving trigonometric equations, some students think of drawing a sine curve but forget about its boundedness. Thus when I ask them to find the maximum and minimum of the function $f(x) = 3\sin 4x$, some can correctly state that it is 3 and -3, respectively. However, if I change the function a little bit, and ask the same question about $g(x) = 3\sin 4x + 4\cos 4x$, then the students either give a wrong answer or try to use a calculator or solve the problem using a derivative, which can work but is not the most efficient method. It would be much better to rewrite the function using an auxiliary argument as $g(x) = 5\sin(4x + \varphi)$ and then state that $-5 \le g(x) \le 5$.

This chapter contains many challenging and Olympiad-type problems and the best methods of solving them. We are going to look at trigonometry not from an application point of view, where knowing nothing more than the relationships in a triangle is needed, but from a different angle. We focus on the unit circle representation of trigonometric functions, their boundedness, restricted domains, and important formulas, along with their derivation. The purpose of this chapter is not only to fill the gaps of the math educational system but also to help you develop a "love" for the subject, so you will be in the "mood to learn" more about trigonometry.

E. Grigorieva, *Methods of Solving Nonstandard Problems*,
DOI 10.1007/978-3-319-19887-3_3

3.1 Introduction to the Unit Circle and Trigonometric Identities

Consider the unit circle (circle of radius 1 with center at the origin O, as shown in Figure 3.1). Let C be the intersection of this circle with the positive X-axis. If t is any real number, we can define the sin(t) and the cos(t) as follows: measure $|t|$ units along the circle from the point C (measuring counterclockwise if $t \geq 0$ and clockwise if $t < 0$). Let $A(x,y)$ be the point on the circle arrived at by thus measuring t units.

Then (by Embry, Calculus, and Linear Algebra) we define its first coordinate as $\cos t$ and the second coordinate as sin(t). Since for any circle the central angle measure equals the radian measure of the corresponding arc, $\angle COA = t$. Hence we have

$$x = \cos(\angle COA) = \cos t$$
$$y = \sin(\angle COA) = \sin t$$

Most trigonometric identities that are so "difficult" to memorize can be seen from the construction on the unit circle. Let us modify Figure 3.1 and reflect point A with respect to the X-axis, obtaining point N and with respect to the vertical axis Y, obtaining point M on the circle and with respect to the origin, obtaining point L. Thus, rectangle $AMLN$ is inscribed into the unit circle and the coordinates of all points are obvious (Figure 3.2).

Using Figure 3.3, first, let us prove that

$$\begin{aligned} \cos(-t) &= \cos t \\ \sin(-t) &= -\sin t \end{aligned} \tag{3.1}$$

The angle $-t$ in Figure 3.3 corresponds to point N on the unit circle, where its first coordinate is the cosine of the corresponding angle and the second coordinate is the sine of the corresponding angle. Since N can be obtained by reflection of A with

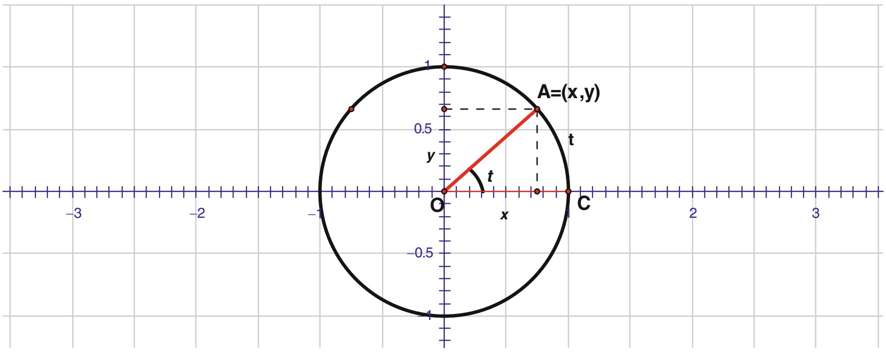


Figure 3.1 Cotangent and Tangent Lines

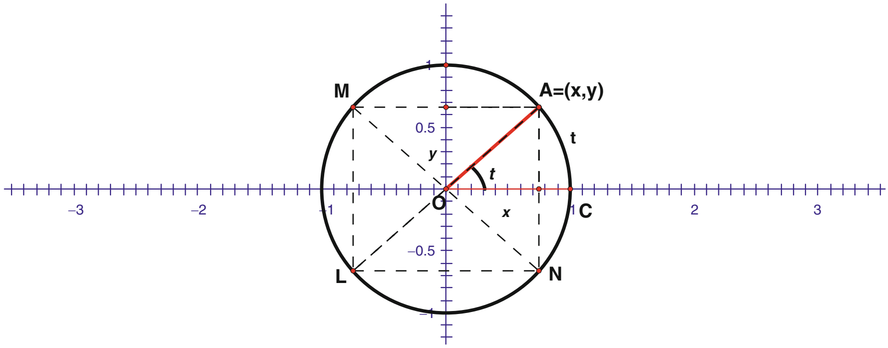


Figure 3.2 AMLN on the unit circle

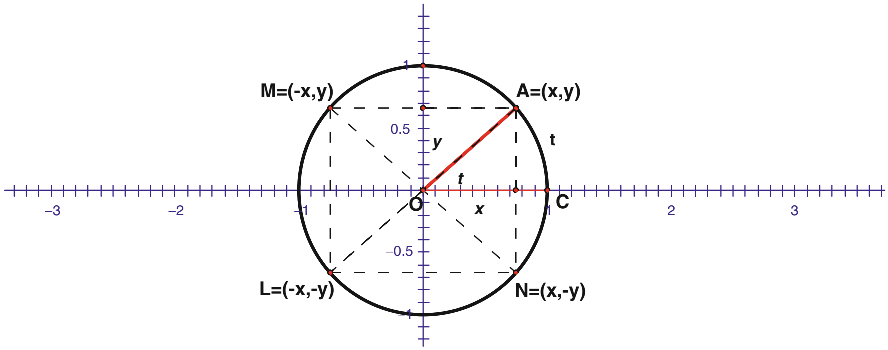


Figure 3.3 Proof of formulas (3.1)–(3.3)

respect of the X axis, points A and N have the same first coordinate ($\cos(-t) = \cos(t)$) and their second coordinates are of opposite sign ($\sin(-t) = -\sin(t)$).

Similarly, using coordinates of points A and M, we can prove these so-called **supplementary identities**:

$$\begin{aligned} \cos(\pi - t) &= -\cos t \\ \sin(\pi - t) &= \ \sin t \end{aligned} \tag{3.2}$$

Point M corresponds to the angle $\angle COM = \pi - t$. Its first coordinate is opposite to that of point A and the second coordinate is the same. The identities are proven.

Next, because points A and L are centrally symmetric with respect to the origin, their coordinates are opposite and their corresponding angles differ by π. Thus if $\angle COA = t \Rightarrow \angle COL = \pi + t$ we can write the following:

$$\begin{aligned} \cos(\pi + t) &= -\cos t \\ \sin(\pi + t) &= -\sin t \end{aligned} \tag{3.3}$$

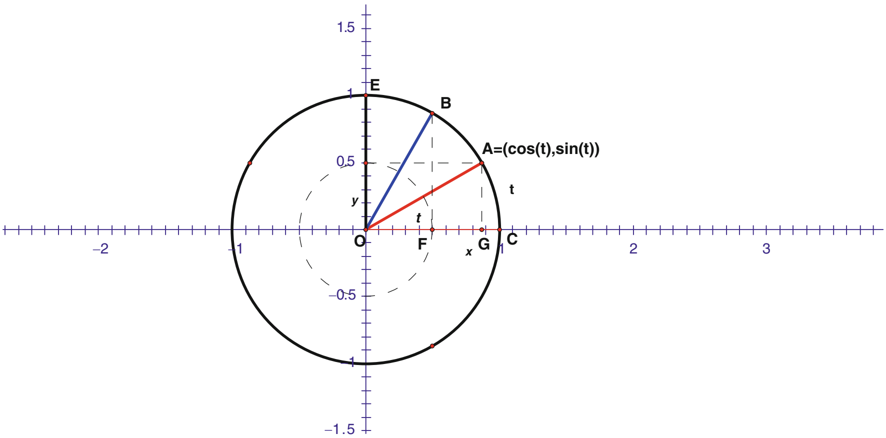


Figure 3.4 Complementary angles

Let us prove the **complementary** angle properties of cosine and sine functions.

Construct point E as the intersection of the unit circle and the vertical axis (Figure 3.4). Measure AG and using a compass make a segment $OF = AG$. For this, we can construct a circle of radius AG at center O until it intersects the horizontal axis at F. Next, through F we will draw a line parallel to OE, so it intersects the circle at B. Since both A and B are on the unit circle $OA = OB = 1$ and from the Pythagorean Theorem for the right triangles OFB and OGA we will obtain that $BF = OG$ and $OF = AG$. Hence the triangles are equal and their corresponding angles are also equal: $\angle COA = \angle OBF = t$ and $\angle FOB = \angle COB = \frac{\pi}{2} - t$. Therefore the following relationships are valid:

$$\begin{aligned} \cos\left(\frac{\pi}{2} - t\right) &= \sin t \\ \sin\left(\frac{\pi}{2} - t\right) &= \cos t \end{aligned} \tag{3.4}$$

or

$$\begin{aligned} \cos(90^\circ - t) &= \sin(t) \\ \sin(90^\circ - t) &= \cos(t) \end{aligned}$$

From this you can always understand why $\cos 30^\circ = \sin 60^\circ$ or $\cos 36^\circ = \sin 54^\circ$.

Let us evaluate some values of the cosine and sine functions. Recall that the circumference of the unit circle is 2π.

Thus, for $t = \frac{\pi}{2}$ (point E on the circle, Figure 3.5) we have coordinates (0, 1), and then $\cos\left(\frac{\pi}{2}\right) = 0$, $\sin\left(\frac{\pi}{2}\right) = 1$. For point $t = \pi$ (point P) we have $\cos(\pi) = -1$, $\sin(\pi) = 0$.

If $t = \frac{5\pi}{2}$, then we again are at the point E (0,1) and algebraically $t = \frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$. Moreover, $\cos t$, $\sin t$ are periodic functions with minimal period $T = 2\pi$ and their values are the same for any values of t that differ by a multiple of a period. Thus,

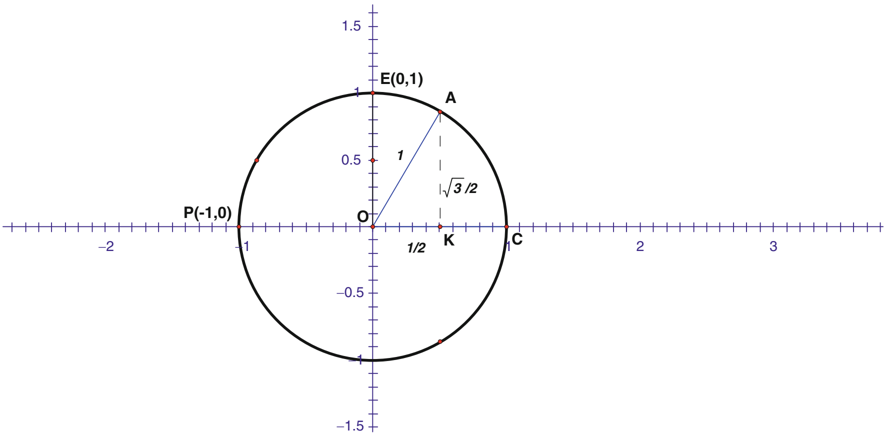


Figure 3.5 Sine and cosine of an angle

$$\begin{aligned} \cos(t + 2\pi \cdot n) &= \cos t \\ \sin(t + 2\pi \cdot n) &= \sin t \end{aligned} \tag{3.5}$$

We can calculate values of sine or cosine for some angles easily.

For example, let $t = \frac{\pi}{3}$(see Figure 3.5). The corresponding point on the unit circle is A. Since the other angle in the triangle formed by dropping a perpendicular line from A to a point K is $\frac{\pi}{6}$ or 30°, in the right triangle OKA: $OK = 1/2$, $AO = 1$, and $KA = \frac{\sqrt{3}}{2}$. Hence, $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

Let us prove the following formulas:

$$\begin{aligned} \cos\left(\frac{\pi}{2} + t\right) &= -\sin t \\ \sin\left(\frac{\pi}{2} + t\right) &= \cos t \end{aligned} \tag{3.6}$$

Draw a unit circle and place a point A corresponding to angle t. Perpendicular to the horizontal axis from A is AM and point B is such that $\angle MOB = t + 90°$ (Figure 3.6). The following will be true:

$$\begin{aligned} &\angle MOA = t,\ \angle MOB = t + \frac{\pi}{2},\ AM = \sin t,\ OM = \cos t \\ &\quad ON = \cos\left(t + \frac{\pi}{2}\right), BN = \sin\left(t + \frac{\pi}{2}\right) \\ &\quad OA = OB = 1 \\ &\angle NOB = 180° - (90° + t) = 90° - t \\ &\angle NBO = t. \end{aligned}$$

Further, because triangles NOB and OMA are congruent by angle-side angle (ASA), their corresponding sides are equal. That can be written as

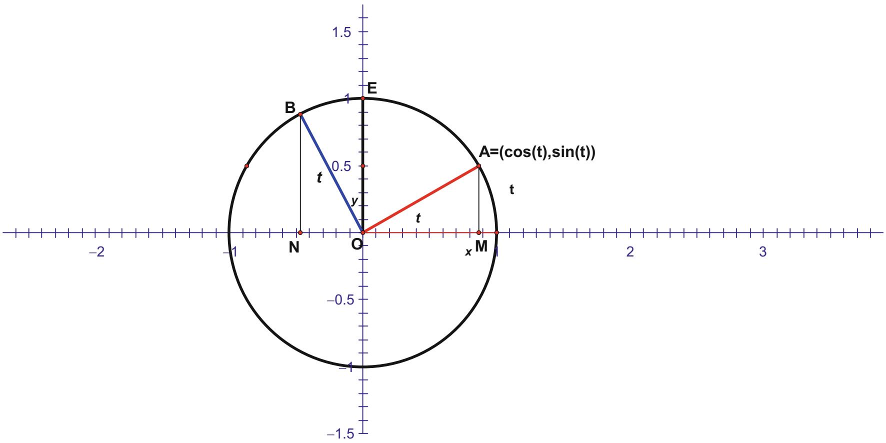


Figure 3.6 Proof of formulas (3.6)

$$\begin{aligned} NO &= AM \\ BN &= OM \end{aligned} \tag{3.7}$$

Since point B is in the second quadrant, its first coordinate is negative. Therefore, (3.7) is the same as (3.6). The proof is completed.

Four other trigonometric functions, tangent (tan), cotangent (cot), secant (sec), and cosecant (csc), can be defined in terms of cosine and sine as follows:

$$\begin{aligned} \tan t &= \frac{\sin t}{\cos t}, & \cos t \neq 0 \\ \cot t &= \frac{1}{\tan t} = \frac{\cos t}{\sin t}, & \sin t \neq 0 \\ \sec t &= \frac{1}{\cos t}, & \cos t \neq 0 \\ \csc t &= \frac{1}{\sin t}, & \sin t \neq 0 \end{aligned} \tag{3.8}$$

Let us make a Table 3.1 of common values of sine, cosine, tangent, secant, and cosecant for some commonly used angles.

3.1.1 Introduction to Inverse Trigonometric Functions

In the following section we will learn methods used to solve simple trigonometric equations. In order to introduce these methods we need to learn about inverse trigonometric functions.

Table 3.1 Common Values of Trigonometric Functions

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–	0	–	0
cot	–	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	–	0	–
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	–	-1	–	1
csc	–	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	–	-1	–

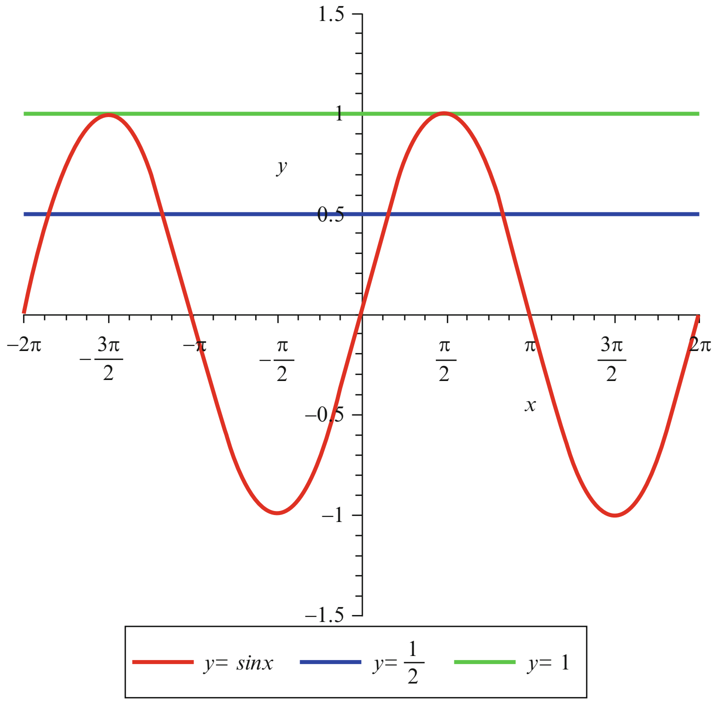


Figure 3.7 Intersections with the sine curve

Consider an equation $\sin x = 1$ or $\sin x = \frac{1}{2}$ that are both geometrically presented in Figure 3.7 on the closed interval $x \in [-2\pi, 2\pi]$.

We understand that on the entire domain, each horizontal line ($y = a$, $-1 \leq a \leq 1$) intersects the sine curve at infinitely many points. In order to describe all these

points, we need to select such subinterval of the domain on which $y = \sin x$ is one to one; that is, there will be only one point of the intersection between the sine curve and horizontal line, $y = a$, $|a| \leq 1$. For example, interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ satisfies this request. There exists a unique value of $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $\sin x = a$.

Arcsine of x is what we call a number $y = \arcsin x$, $-1 \leq x \leq 1$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ such that $\sin y = x$. Thus, a function $y = \arcsin x$ is an inverse to $y = \sin x$ with domain: $-1 \leq x \leq 1$ and range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Because the graphs of the inverses are symmetric with respect to the line $y = x$, the function $y = \sin x$ can have its inverse only on the closed interval $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ on which it is a one-to-one function (the function is monotonically increasing on this interval). Therefore, the range of $y = \arcsin x$ is restricted as $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Both functions go through the origin, have central symmetry, and are odd functions:

$$\arcsin(-x) = -\arcsin x. \tag{3.9}$$

For example, $\arcsin\left(-\frac{1}{2}\right) = -\arcsin\frac{1}{2} = -\frac{\pi}{6}$. The graphs of both functions are shown in Figure 3.8.

Arccosine of x is defined as follows: $y = \arccos x$, $-1 \leq x \leq 1$, $0 \leq y \leq \pi$ such that $\cos y = x$.

Thus the function $y = \arccos x$ is the inverse to $y = \cos x$ with domain: $-1 \leq x \leq 1$ and range: $0 \leq y \leq \pi$. (In this case, the function $y = \cos x$ is one to one only on the interval $x \in [0, \pi]$ and on this interval it has its inverse.) (See Figure 3.10.) This function is neither odd nor even. However, it is central symmetric to the point $\left(0, \frac{\pi}{2}\right)$ and this is why the following is true:

$$\arccos(-x) = \pi - \arccos x. \tag{3.10}$$

In order to evaluate $\arccos(p), |p| \leq 1$, we can use the unit circle idea.

For example, what is $\arccos\left(-\frac{1}{2}\right)$? Many students wrongly believe that it has the same value as $\arccos\frac{1}{2} = \frac{\pi}{3}$. However, inverse cosine is not an even function, and $\arccos\left(-\frac{1}{2}\right)$ must equal the angle from the interval $0 \leq \alpha \leq \pi$ (marked by the red arc on the unit circle, Figure 3.9), the cosine of which is $-1/2$, and this would be the angle $\frac{2\pi}{3}$.

Arctangent of x is defined as follows: $y = \arctan x$, $x \in R$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$ such that $\tan y = x$. The function $y = \arctan x$ is inverse to $y = \tan x$ with the domain: $x \in R$ and range: $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (Figure 3.11). It is an odd function; therefore,

$$\arctan(-x) = -\arctan x \tag{3.11}$$

Arccotangent of x is defined as follows: $y = \operatorname{arccot} x, x \in R$, $0 < y < \pi$ such that $\cot y = x$. The function $y = \operatorname{arc\,cot} x$ is inverse to $y = \cot x$ with domain: $x \in R$ and range: $y \in (0, \pi)$. It is centrally symmetric with respect to point $\left(0, \frac{\pi}{2}\right)$ (Figure 3.12).

It will be useful to remember the following formulas:

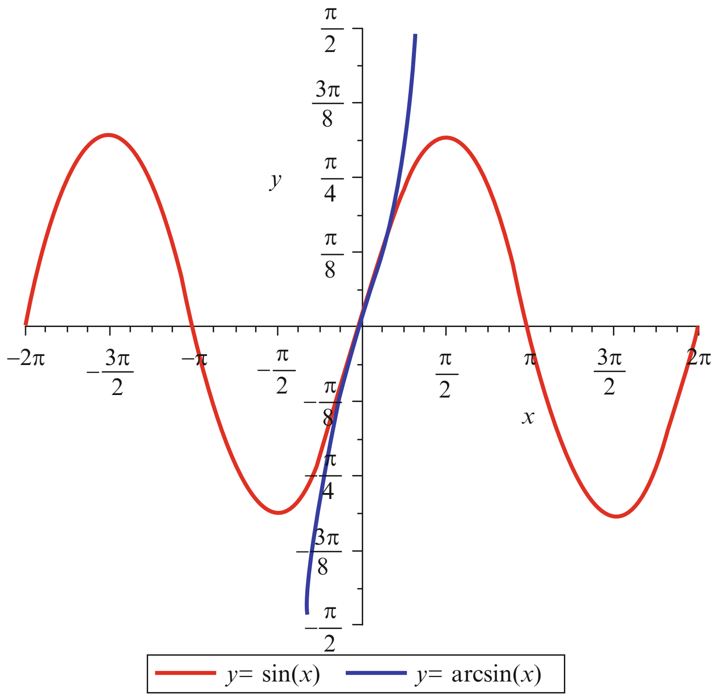


Figure 3.8 The graphs of $y = \sin x$ and $y = \arcsin x$

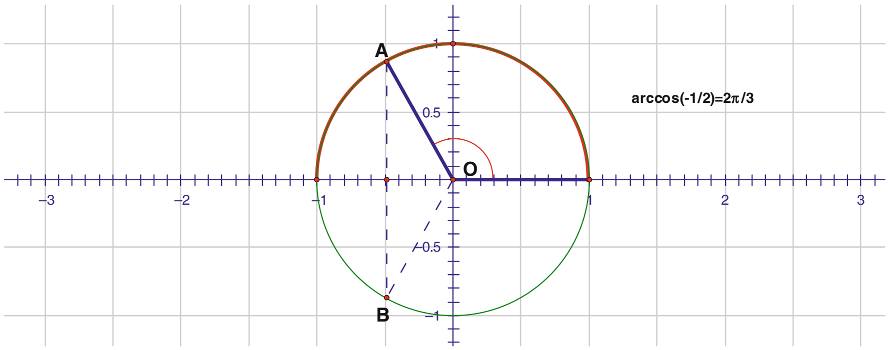


Figure 3.9 Finding $\arccos\left(-\frac{1}{2}\right)$

$$\begin{aligned} &\sin(\arcsin a) = a,\ \cos(\arccos a) = a,\ |a| \leq 1 \\ &\tan(\arctan b) = b,\ \cot(\operatorname{arccot} b) = b, b \in R \end{aligned} \tag{3.12}$$

$$\arctan a + \operatorname{arccot} a = \frac{\pi}{2} \tag{3.13}$$

$$\arcsin x + \arccos x = \frac{\pi}{2} \tag{3.14}$$

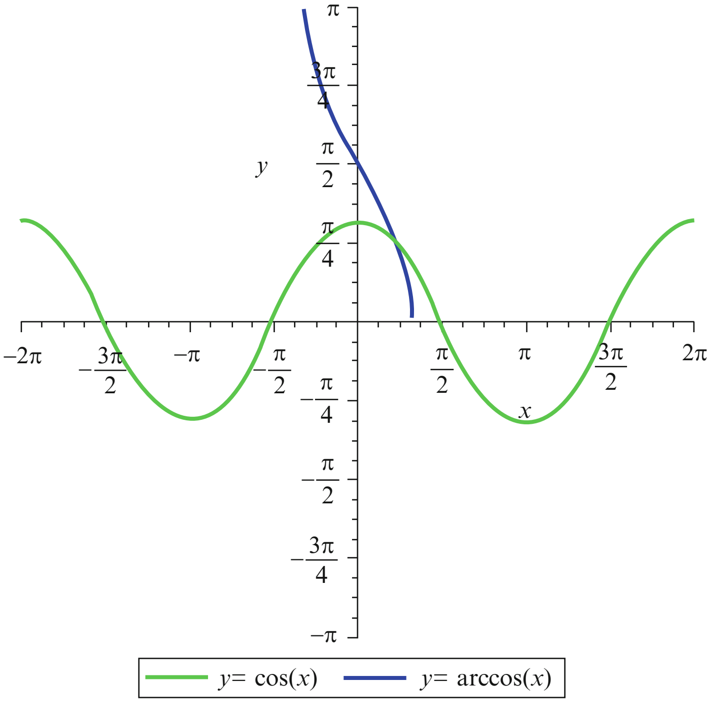


Figure 3.10 Graphs of $y = \cos x$ and $y = \arccos x$

Problem 97 Evaluate $\arccos\left(\sin\left(-\frac{\pi}{9}\right)\right)$.

Solution It follows from the definition of arccosine that

$$\arccos(\cos x) = x, \quad \text{if } 0 \leq x \leq \pi \tag{3.15}$$

Hence, in order to use this formula we need to replace the value of $\sin\left(-\frac{\pi}{9}\right)$ by the cosine of an angle concluded between 0 and π. Using the complementary angle formula, we have

$$\sin\left(-\frac{\pi}{9}\right) = -\sin\left(\frac{\pi}{9}\right) = \cos\left(\frac{\pi}{2} + \frac{\pi}{9}\right) = \cos\frac{11\pi}{18}$$
$$0 \leq \frac{11\pi}{18} \leq \pi$$

Therefore, $\arccos\left(\sin\left(-\frac{\pi}{9}\right)\right) = \arccos\left(\cos\left(\frac{11\pi}{18}\right)\right) = \frac{11\pi}{18}$

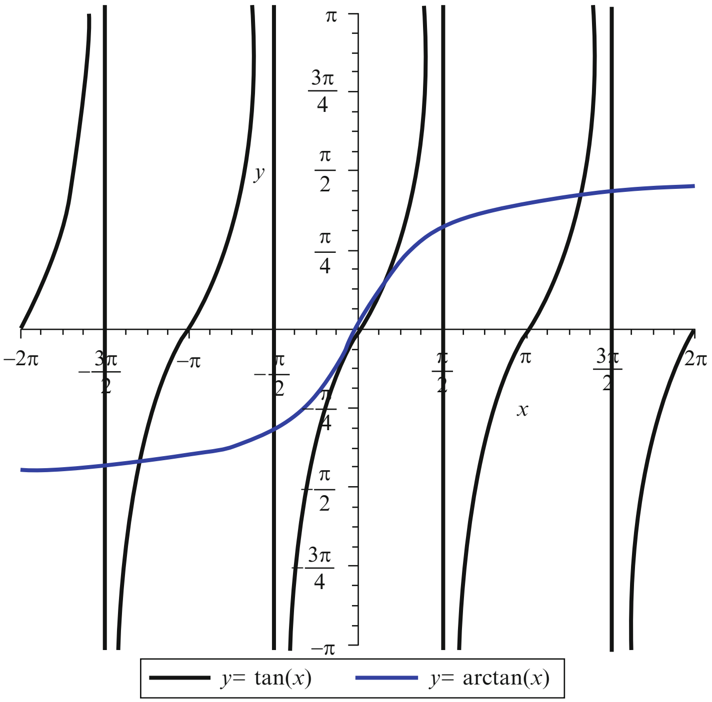


Figure 3.11 Graphs of $y = \tan x$ and $y = \arctan x$

Answer $\dfrac{11\pi}{18}$.

Problem 98 Evaluate $\arcsin\left(\cos\left(\dfrac{31\pi}{5}\right)\right)$.

Solution It follows from the definition of arcsine that

$$\arcsin(\sin x) = x, \quad \text{if} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \tag{3.16}$$

Using an approach similar to that of the previous problem we will rewrite cosine in terms of sine of the angle concluded in the interval above:

$$\cos\frac{31\pi}{5} = \cos\left(6\pi + \frac{\pi}{5}\right) = \cos\frac{\pi}{5} = \sin\left(\frac{\pi}{2} - \frac{\pi}{5}\right) = \sin\frac{3\pi}{10}$$

Therefore, $\arcsin\left(\cos\left(\dfrac{31\pi}{5}\right)\right) = \arcsin\left(\sin\left(\dfrac{3\pi}{10}\right)\right) = \dfrac{3\pi}{10}$.

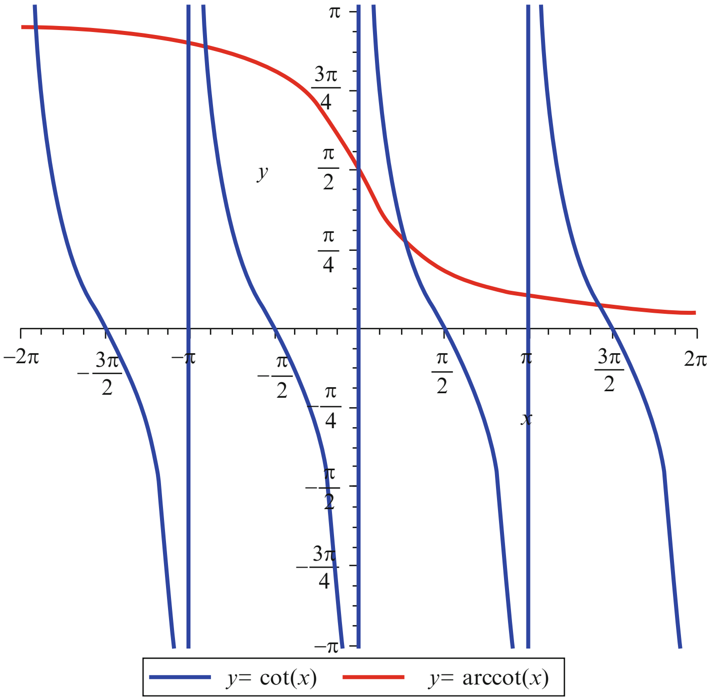


Figure 3.12 Graphs of $y = \cot x$ and $y = \operatorname{arc}\cot x$

Answer $\dfrac{3\pi}{10}$.

Problem 99 Solve the equation $(\arcsin x)^3 + (\arccos x)^3 = \alpha\pi^3$ for all $\alpha \in R$.

Solution In Chapter 2 we learned formula (2.16)

$$a^3 + b^3 = (a + b)^3 - 3ab(a + b)$$

Applying this formula and formula (3.14), and after simplification, the equation can be written as

$$12y^2 - 6\pi y + (1 - 8\alpha)\pi^2 = 0, \text{ where } y = \arcsin x.$$

Solutions of a quadratic equation depend on its discriminant. We can use D/4 formula because the coefficient of the linear term is even:

$$\frac{D}{4} = (3\pi)^2 - (1 - 8\alpha)12\pi^2$$
$$\frac{D}{4} = 9\pi^2 - 12\pi^2(1 - 8\alpha)$$
$$\frac{D}{4} = 3\pi^2(32\alpha - 1)$$

There are three possible outcomes:

1. If $\alpha < \frac{1}{32}$ there are no real solutions.
2. If $\alpha = \frac{1}{32}$ then D/4 = 0 and we have one solution

$$y = \frac{3\pi^2}{12\pi} = \frac{\pi}{4}$$
$$\arcsin x = \frac{\pi}{4}$$
$$x = \frac{\sqrt{2}}{2}$$

3. If $\alpha > \frac{1}{32}$ then there are two real solutions for the quadratic equation in y

$$y = \frac{3\pi^2 \pm \sqrt{3\pi^4(32\alpha - 1)}}{12\pi}$$
$$y = \frac{\pi}{12} \cdot \left(3 \pm \sqrt{3(32\alpha - 1)}\right)$$
$$\left[\begin{array}{l} \arcsin x = \frac{\pi}{12} \cdot \left(3 + \sqrt{3(32\alpha - 1)}\right) \\ \arcsin x = \frac{\pi}{12} \cdot \left(3 - \sqrt{3(32\alpha - 1)}\right) \end{array}\right.$$

Because arcsine changes on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we have to investigate if there are any other restrictions on α. Thus the following must be true:

$$-\frac{\pi}{2} \le \frac{\pi}{12}\left(3 + \sqrt{96\alpha - 3}\right) \le \frac{\pi}{2}$$
$$\frac{\pi}{12}\left(3 + \sqrt{96\alpha - 3}\right) \le \frac{6\pi}{12}$$
$$3 + \sqrt{96\alpha - 3} \le 6$$
$$\sqrt{96\alpha - 3} \le 3$$
$$96\alpha - 3 \le 3^2$$
$$96\alpha \le 12$$
$$\alpha \le \frac{1}{8}$$

and

$$
\begin{aligned}
&-\frac{\pi}{2} \le \frac{\pi}{12} \cdot \left(3 - \sqrt{96\alpha - 3}\right) \le \frac{\pi}{2} \\
&-6 \le \left(3 - \sqrt{96a - 3}\right) \le 6 \\
&-9 \le -\sqrt{96a - 3} \le 3 \\
&-3 \le \sqrt{96a - 3} \le 9 \\
&96a - 3 \le 81 \\
&96a \le 84 \\
&a \le \frac{7}{8}
\end{aligned}
$$

Putting together the two restriction on the parameter we finally obtain the following solution to case 3:

If $\frac{1}{32} < \alpha < \frac{1}{8}$ then $x = \sin\left(\frac{\pi}{12}\left(3 \pm \sqrt{96\alpha - 3}\right)\right)$

If $\alpha = \frac{1}{8}$ then

$$
x = \sin\left(\frac{\pi}{12}(3 \pm 3)\right)
$$

$$
\left[\begin{array}{l} x = \sin\frac{\pi}{2} = 1 \\ x = \sin 0 = 0 \end{array}\right.
$$

Answer If $\alpha < 1/32$, then there are no real solutions.

If $\alpha = \frac{1}{32}$ then $x = \frac{1}{\sqrt{2}}$.

If $\frac{1}{32} < \alpha < \frac{1}{8}$, then $x = \sin\left(\frac{\pi}{12}\left(3 \pm \sqrt{96\alpha - 3}\right)\right)$,

If $\alpha = \frac{1}{8}$, then $x = 0, \quad x = 1$.

3.2 Best Methods for Solving Simple Equations and Inequalities

In this section I demonstrate the most efficient methods of solving simple trigonometric equations.

3.2.1 *Solving* $\cos x = a$

Rephrase the problem: Find points on the unit circle, for which the first coordinate equals a. This geometric approach will help you to find the solution right away.

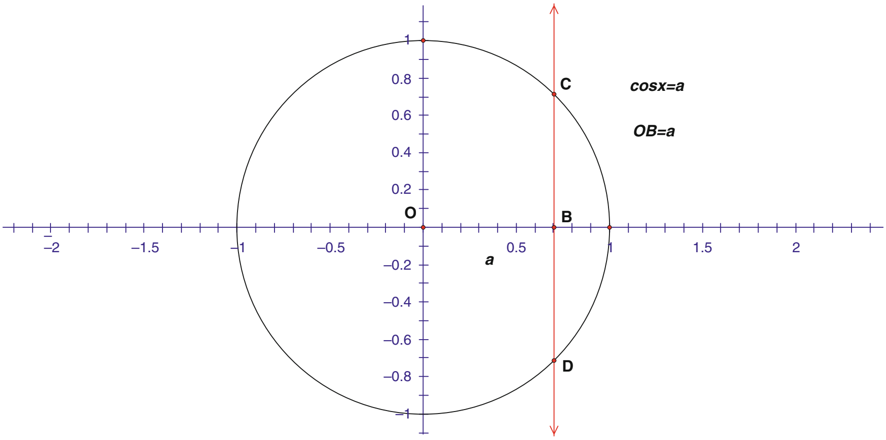


Figure 3.13 Vertical line and the unit circle

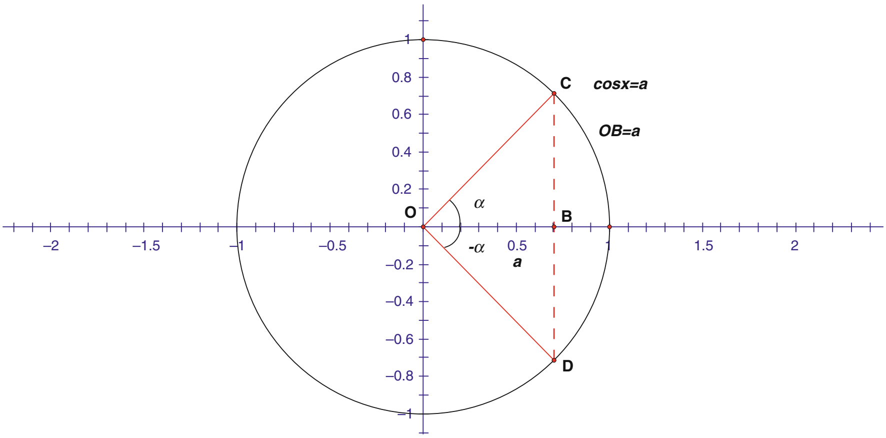


Figure 3.14 Solving $\cos x = a$

Draw a picture and for simplicity, assume that a is a positive real number less than or equal 1 (Figure 3.13).

If OB $= a$ then there are two points (C and D) within each revolution on the unit circle that have a as their first coordinate. By finding the angles corresponding to these points we will find the solution of the problem! (Figure 3.14)

By connecting O and C and O and D, we find that point C corresponds to $\angle BOC = \alpha$ and point D (as symmetric to C) to $\angle BOD = -\alpha$. The angle $\alpha = \arccos a$, then $-\alpha = -\arccos a$.

Uniting these two answers and applying periodicity of the cosine function, the solution is

$$\begin{aligned} &\cos x = a, \ |a| \le 1 \\ &x = \pm \arccos a + 2\pi \cdot n, \quad n \in Z. \end{aligned} \tag{3.17}$$

Here and below, the notation $n \in Z$ means that $n = 0, \pm 1, \pm 2, \ldots$

If the value of a is one corresponding to the cosine row of Table 3.1, then arccos a is the simplest corresponding angle between 0 and π. For example, if $a = \frac{1}{2} \Rightarrow \arccos\frac{1}{2} = \frac{\pi}{3}$ or $a = -1 \Rightarrow \arccos(-1) = \pi$, etc.

Many trigonometric equations are not that simple and usually after solving them we will need to write their solutions in the simplest form. Consider some common equations, positions of points, and solutions. We will learn how to describe points on the unit circle based on their location on the circle.

Problem 100 Solve the equation $\cos^2 x = \dfrac{3}{4}$.

Solution This simple equation can be written as

$$\cos x = \pm \frac{\sqrt{3}}{2}$$

$$\left[\begin{aligned} &x = \pm\frac{\pi}{6} + 2\pi n \\ &x = \pm\frac{5\pi}{6} + 2\pi m, \quad n, m \in Z. \end{aligned}\right.$$

The first solution is given by points B and Q and the second one by points E and M on the unit circle (Figure 3.15).

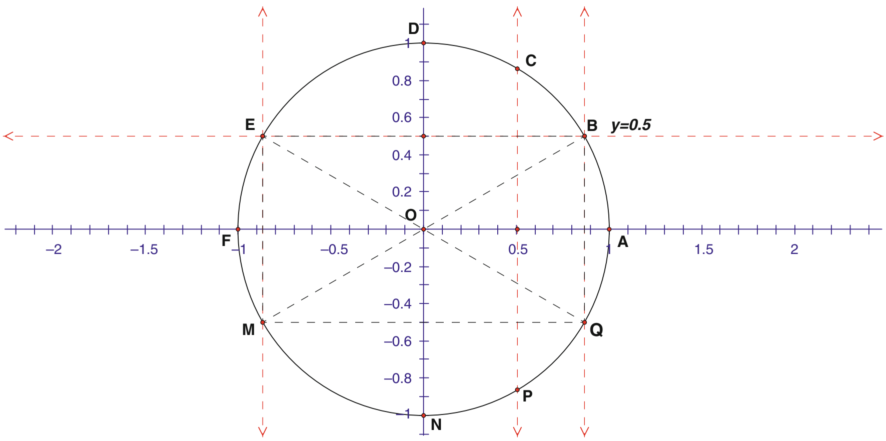


Figure 3.15 Sketch for Problem 100

Can this answer be simplified or written in a better form? Yes. By connecting points B and M and Q and E, we can see that these corresponding pairs are symmetric with respect to the origin, and their angles differ by π.

Answer $x = \pm\frac{\pi}{6} + \pi \cdot k, \quad k \in Z.$

Using Figure 3.15, we will solve the following simple problems that will help you learn to represent your answer in the simplest and most efficient form.

Problem 101 Describe the angle that corresponds to point B shown in Figure 3.15.

Solution Second coordinate of point B is ½, so B corresponds to $t = \frac{\pi}{6} + 2\pi n, n \in Z$.

Problem 102 The solution of some trigonometric equation is given by points B and Q on the unit circle (Figure 3.15). Find that solution.

Solution Points B and Q are symmetric with respect to the X-axis; then using directed angles $\angle AOB = \frac{\pi}{6} \Rightarrow \angle AOQ = -\frac{\pi}{6}$.

Both points together can be described by $t = \pm\frac{\pi}{6} + 2\pi n, \quad n \in Z$.

Problem 103 The solution of some trigonometric equation is given by two points B and M on the unit circle shown in Figure 3.15 above. Describe the solution.

Solution Points B and M are symmetric with respect to the origin. The angles corresponding to them differ by π. Thus, the position of M can be written as $t = \frac{\pi}{6} + \pi = \frac{7\pi}{6}$. However, if we add another π to this angle, we would jump to point B again, and by adding one more π, we would again be at M. The formula that gives us this alternation is

$$t_{B-M} = \frac{\pi}{6} + \pi n, \quad n \in Z.$$

Problem 104 Now assume that a solution is given by four points on the circle B, E, M, and Q. How can you describe all these points together?

Solution Points $\{B$ and $E\}$ and $\{M$ and $Q\}$ are symmetric to the Y-axis and $\{B, M\}$ and $\{E,Q\}$ are centrally symmetric. Therefore, all four points can be described by

$$t_{B-E-M-Q} = \pm\frac{\pi}{6} + \pi n, \quad n \in Z.$$

If you are in doubt, please try the first four values of n. For example, if

$$\begin{aligned} n = 0, \quad & t = \pm\frac{\pi}{6} && \Leftrightarrow \quad t = \frac{\pi}{6}, \quad t = -\frac{\pi}{6} \\ n = 1, \quad & t = \pm\frac{\pi}{6} + \pi && \Leftrightarrow \quad t = \frac{7\pi}{6}, \quad t = \frac{5\pi}{6} \end{aligned}$$

The first pair of the answers describes points B and Q, respectively. The second pair of answers describes points M and E, respectively.

Problem 105 Describe the solution given by a) points D, N, F, and A; b) by the pairs D and N and then by F and A; and then c) by all four points together (Figure 3.15).

Solution

a) Point D has coordinates (0, 1) and corresponds to the smallest angle $\frac{\pi}{2}$. There will be infinitely many angles that correspond with this point, all of them given by $t_D = \frac{\pi}{2} + 2\pi n$.

   Point N is symmetric to D with respect to the origin; hence their angles differ by π. It can be described by $t_N = \frac{3\pi}{2} + 2\pi n$ or by $t_N = -\frac{\pi}{2} + 2\pi m$.

   Point A has coordinates (1, 0) and matches with angle 0 or in general with $t_A = 2\pi n, \ n \in Z$.
b) If we want to describe two points together, then D and N are associated with $t_{D-N} = \frac{\pi}{2} + \pi n, \ n \in Z$.
c) All four points $\{A, D, F, N\}$ can be seen together and the difference between locations of two consecutive points is the same, $\frac{\pi}{2}$. Therefore, all points can be described by one formula $t_{A-D-F-N} = \frac{\pi}{2} \cdot n, \quad n \in Z$.

Problem 106 Describe points M and Q together (Figure 3.15).

Solution Points M and Q are symmetric with respect to the Y-axis, and they can be described by supplemental angles. Hence, if M can be described by angle α, then Q would be $\pi - \alpha$. Both points together can be described by $t = (-1)^n$ $\arcsin\left(-\frac{\pi}{6}\right) + \pi n, n \in Z$. This formula is often used when solving the equation $\sin x = a$.

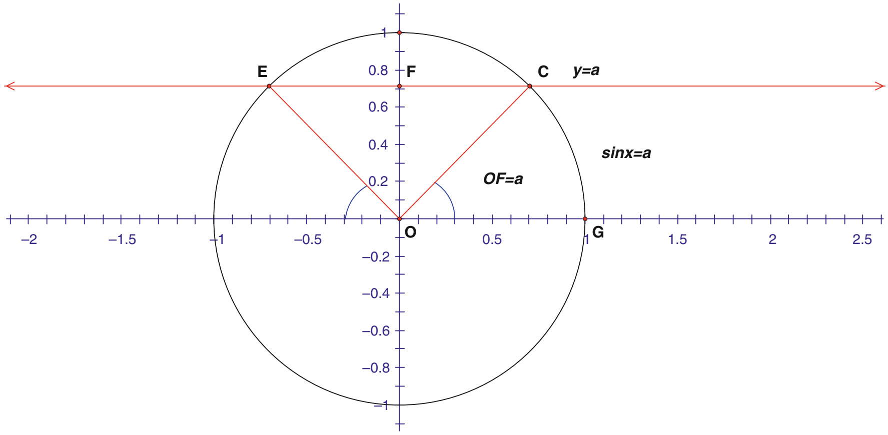


Figure 3.16 Solving $\sin x = a$

3.2.2 Solving sinx = a

For simplicity, assume that $0 \le a \le 1$. We are looking for such points on the unit circle that have the second coordinate a. There are two points on the circle that satisfy this condition $\{E$ and $C\}$. Look at the sketch below (Figure 3.16). The points are at the intersection of line $y = a$ and the unit circle.

The smallest angle corresponding to C $\angle GOC$ is given by $\arcsin a$; the other angle associated with E is $\pi - \arcsin a$. (From symmetry, we know that the angles between EO and the X-axis and CO and the X-axis are the same.)

Hence, the solution to $\sin x = a$ can be written as

$$\begin{aligned} &\sin t = a, \\ &t = (-1)^n \arcsin a + \pi n, \quad n \in Z. \end{aligned} \tag{3.18}$$

For example, if you need to solve the following problem:

Problem 107 Find all $x \in \left[\frac{\pi}{2}, \frac{11\pi}{2}\right]$ satisfying the equation $\sin x = \frac{1}{2}$.

Solution First, we will draw a unit circle and since sine is the second coordinate of a point on the unit circle, we will find such points by drawing a horizontal line through point D with y coordinate ½ until the line intersects the circle at two points, B and A (Figure 3.17).

Both points A and B correspond to the value of sine of ½; hence we need to find their angles. Within each revolution, point A matches with angle

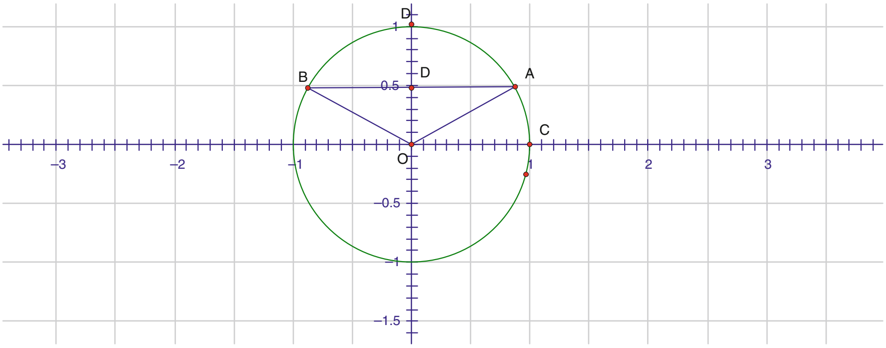


Figure 3.17 Solving $\sin x = \frac{1}{2}$

$\angle COA = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ and point B with angle $\angle COB = \pi - \arcsin\left(\frac{1}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Therefore, the general solution to the given equation can be written as

$$\left[\begin{array}{l} x = \dfrac{\pi}{6} + 2\pi \cdot n, \\ x = \dfrac{5\pi}{6} + 2\pi \cdot n, \quad n = 0, \pm 1, \pm 2, \ldots \end{array}\right.$$

However, we need to select only solutions $x \in I = \left[\frac{\pi}{2}, \frac{11\pi}{2}\right]$.

How can we select all valid values of the variable x?

Let us find out how far down by integer n we should go.

Since $5\pi = \frac{10\pi}{2} < \frac{11\pi}{2} < \frac{12\pi}{2} = 6\pi = 2\pi \cdot 3$, we will consider only such n that satisfy $0 \le n \le 2$.

It is clear that if $n = 0$, then $x = \frac{\pi}{6} \notin I$ and that $x = \frac{5\pi}{6} \in I$.

At $n = 1$ both solutions $x = \frac{\pi}{6} + 2\pi = \frac{13\pi}{6} \in I$ and $x = \frac{5\pi}{6} + 2\pi = \frac{17\pi}{6} \in I$.

Further, if $n = 2$ then $x = \frac{\pi}{6} + 2\pi \cdot 2 = \frac{25\pi}{6} \in I$ and

$x = \frac{5\pi}{6} + 4\pi = \frac{29\pi}{6} < 5\pi < \frac{33\pi}{6} = \frac{11\pi}{2} \in I$.

Therefore, we have only five values of x that satisfy the condition of the problem.

Answer $\left\{\frac{5\pi}{6}; \frac{13\pi}{6}; \frac{17\pi}{6}; \frac{25\pi}{6}; \frac{29\pi}{6}\right\} \in \left[\frac{\pi}{2}, \frac{11\pi}{2}\right]$.

3.2.3 Solving tanx = a and cotx = a

We will draw a unit circle and a so-called **tangent line**—a vertical line that is tangent to the circle at point B (1,0). Positive values of a will be above B and negative below (Figure 3.18). Thus, $\tan x = 0$ corresponds to point B. In order to solve $\tan x = a$, we will place value of a on the tangent line. Let $a > 0$, for example, $BG = a$, and connect G with the origin until it would intersect the unit circle at two points (in our sketch at E and F). Yes, there are two angles $\angle BOE$ and $\angle BOF$ on the unit circle corresponding to the given positive value of a. From triangle GOB we

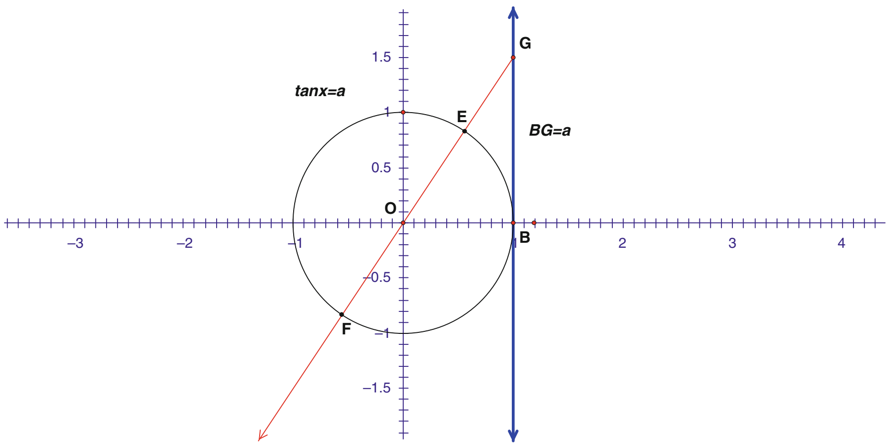


Figure 3.18 Tangent line

have $OB = 1, BG = a \Rightarrow \tan(\angle BOG) = \tan(\angle BOE) = a$. The angle x is arctangent of a.

Points E and F differ by π. Hence, if the angle $\angle BOE = \arctan a$, then $\angle BOF = \pi + \arctan a$.

Both angles satisfy the equation. Therefore this can be written as

$$\begin{aligned} &\tan x = a \\ &x = \arctan a + \pi n, \quad n \in Z \end{aligned} \tag{3.19}$$

Example Solve the equation $\tan\left(\frac{\pi}{4} - 3x\right) = 1$.

Solution Since tangent is an odd function, the equation can be rewritten as

$$\begin{aligned} &\tan\left(3x - \frac{\pi}{4}\right) = -1 \\ &3x - \frac{\pi}{4} = -\frac{\pi}{4} + \pi n \\ &3x = \pi n \\ &x = \frac{\pi}{3} \cdot n, \ n \in Z. \end{aligned}$$

Solving $\cot x = a$ is similar to solving $\tan x = a$. We need to draw a unit circle and **a cotangent line** that is tangent to the circle at point (0,1). Cotangent value is zero at this point, positive to the right of this point and negative to the left.

In Figure 3.19, you can see both cotangent and tangent lines. Also if the angle t is given then the values of $\tan t$ (segment CD) and $\cot t$ (segment AB) are both shown in Figure 3.19.

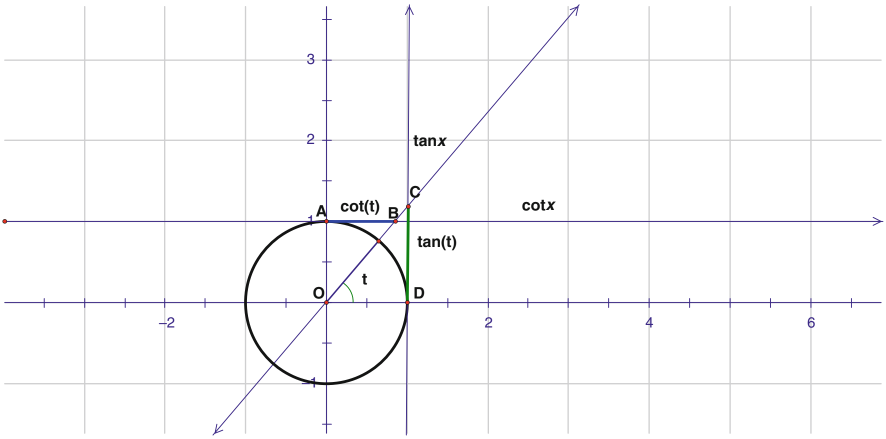


Figure 3.19 Cotangent line

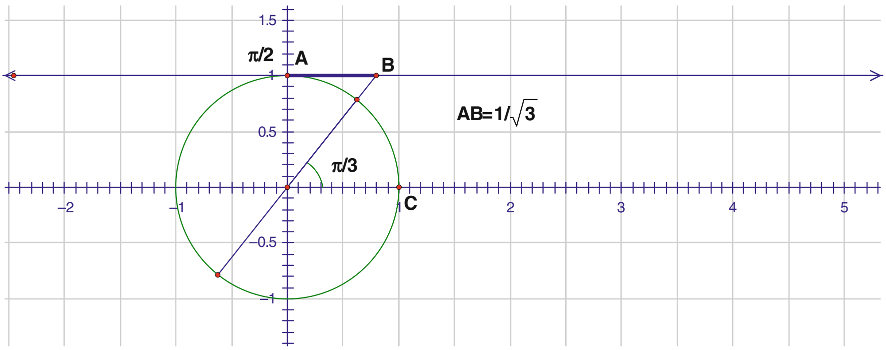


Figure 3.20 Using cotangent line

In general, the following is true:

$$\begin{aligned} &\cot x = a \\ &x = \operatorname{arc}\cot a + \pi n, \quad n \in Z \end{aligned} \tag{3.20}$$

In Figure 3.20, we showed the solution to the equation $\cot x = \frac{1}{\sqrt{3}}$. There are two points on the unit circle that correspond to this value of the cotangent. Both can be written as $x = \frac{\pi}{3} + \pi n, \quad n \in Z.$

3.2.4 Solving Trigonometric Inequalities

Many problems in trigonometry require selecting a solution from a given interval. This can be asked directly as in the following problem:

Solve the equation

$$\left(1+\tan^2 x\right)\sin x-\tan^2 x+1=0$$

Subject to the inequality $\tan x>0$.

Or indirectly when we have to be careful in selecting a correct solution. For example, in problem:

$$\sqrt{1-2\sin^2 x}=\sqrt{\sin x}.$$

In solving this problem, because of the square roots, we need to consider only the solutions that will satisfy the following system:

$$\begin{cases} 1-2\sin^2 x\geq 0 \\ \sin x\geq 0 \\ \sin x=1-2\sin^2 x \end{cases}$$

It is useful to be able to sketch the solutions of the inequalities on the unit circle.

Below we will consider some simple examples, so you can understand how to find geometric solutions to trigonometric inequalities:

Example 1 $\sin x>\frac{1}{2}$

Example 2 $\cos x<\frac{1}{2}$

Example 3 $\begin{cases} \sin x\geq -\dfrac{1}{2} \\ \cos x\leq \dfrac{1}{2} \end{cases}$

Example 4 $\cot x<\frac{2}{3}$

Example 5 $\tan^2 x>\frac{1}{3}$

Example 1 Draw the unit circle (Figure 3.21).
There are two points corresponding to $\sin x=\frac{1}{2}$.

The angles satisfying the inequality $\sin x>\frac{1}{2}$ are marked by the thick green arc and algebraically can be described as $\frac{\pi}{6}+2\pi n<x<\frac{5\pi}{6}+2\pi n,\ n\in Z$.

Remember that each solution arc on the unit circle must be described by an interval between two angles, from the smallest to the biggest, by moving counter-clockwise along the arc. For example, if we instead started from the left point on the same circle and gave the following interval, $\frac{5\pi}{6}+2\pi n<x<\frac{13\pi}{6}+2\pi n$, then it would describe the solution to the opposite inequality, $\sin x<\frac{1}{2}$, and would be given by the bigger arc.

Example 2 is shown in Figure 3.22 and its solution is marked by the red arc ABC. Algebraically it can be written as $\frac{\pi}{3}+2\pi n<x<\frac{5\pi}{3}+2\pi n,\ \ n\in Z$.

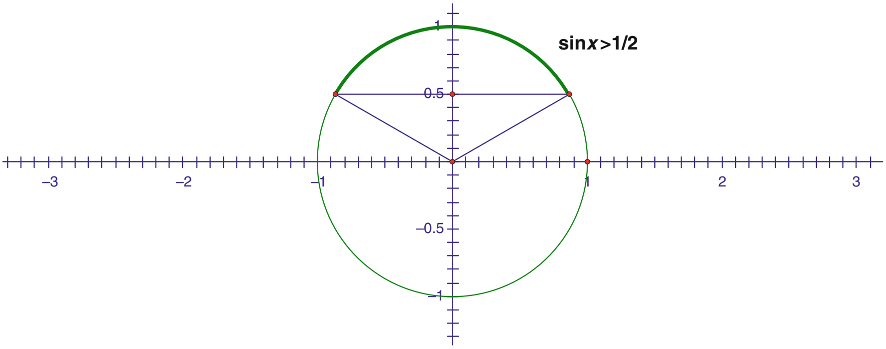


Figure 3.21 Sketch for Example 1

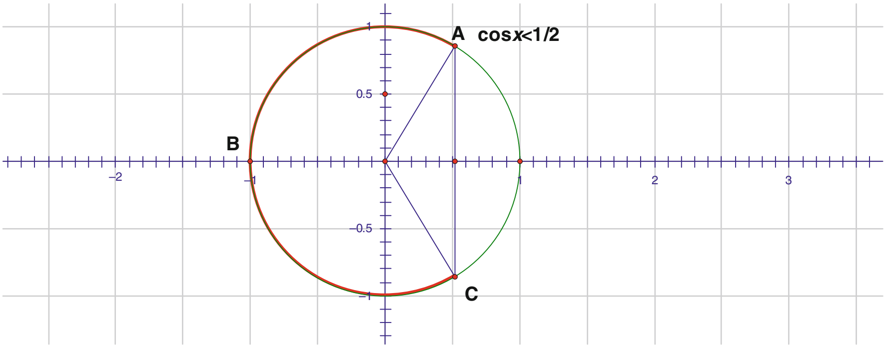


Figure 3.22 Sketch for Example 2

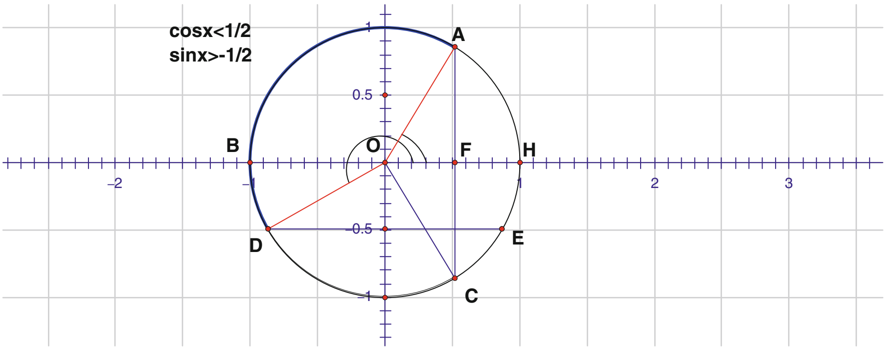


Figure 3.23 Sketch for Example 3

For *Example 3* (system), each inequality must be sketched separately and then the intersection (blue arc *ABD*) must be found (see Figure 3.23).

Thus, solution to the first inequality, $\cos x \le \frac{1}{2}$, is given by the arc *ABC*. The solution to $\sin x \ge -\frac{1}{2}$ is given by the arc *EBD*. The common part of the two arcs is

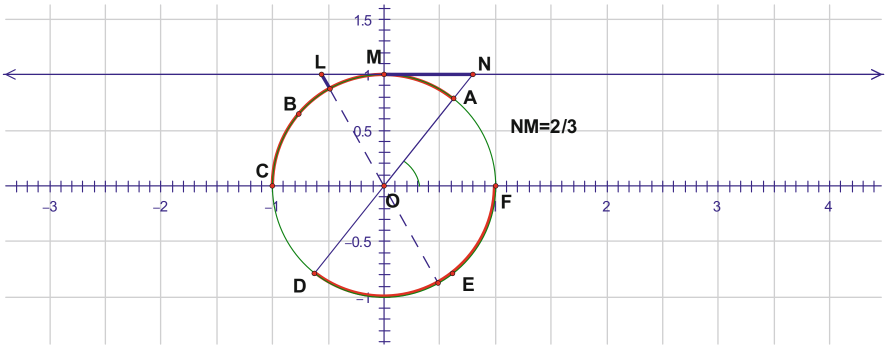


Figure 3.24 Sketch for Example 4

arc *ABD*. The answer lies between angles, including the end points of the arcs, and it can be written as $\frac{\pi}{3}+2\pi n\leq x\leq\frac{7\pi}{6}+2\pi n$.

Let us solve *Example 4*, $\cot x<\frac{2}{3}$.

Draw the unit circle and cotangent line that is tangent to the circle at point *M* (0,1). Mark 2/3 on the positive part of the line (segment *NM*) and connect point N with the center of the circle until it intersects the circle again at point *D* (Figure 3.24).

The solution to the equation $\cot x=\frac{2}{3}$ will be given by these two points and can be written as $x=\mathrm{arccot}\frac{2}{3}+\pi n$.

Next, we will try to visualize what would happen to the solution if we start moving point *N* to the left on the cotangent line and obtain a solution by connecting a new point (for example, *L*) with the origin again in the same manner. It is clear that there are two arcs for which the inequality is true, *ABC* and *DEF* (Figure 3.24).

Finally, our solution can be written as

$$\mathrm{arccot}\frac{2}{3}+\pi n<x<\pi+\pi n,\quad n\in Z.$$

In order to find a solution to *Example 5* and solve $\tan^2 x>\frac{1}{3}$, first, it is convenient, using the difference of squares formula, to rewrite the inequality as follows: $\left(\tan x-\frac{1}{\sqrt{3}}\right)\left(\tan x+\frac{1}{\sqrt{3}}\right)>0$. Then the solution is possible when either both quantities inside parentheses are positive or both negative. This can be written as

$$\left[\begin{array}{l}\left\{\begin{array}{l}\tan x>\dfrac{1}{\sqrt{3}}\\ \tan x>-\dfrac{1}{\sqrt{3}}\end{array}\right.\\ \left\{\begin{array}{l}\tan x<\dfrac{1}{\sqrt{3}}\\ \tan x<-\dfrac{1}{\sqrt{3}}\end{array}\right.\end{array}\right.$$

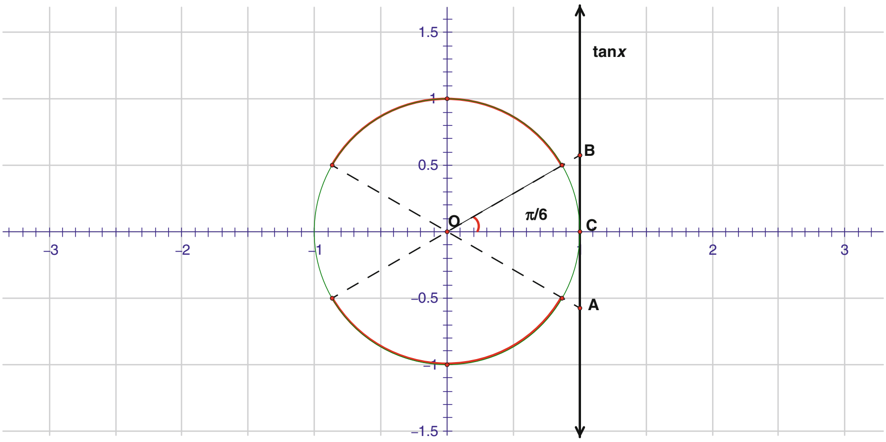


Figure 3.25 Solving the system of inequalities

Finally, the union of the two systems is given graphically in Figure 3.25. Algebraically it can be written as

$$\frac{\pi}{6} + \pi n < x < \frac{5\pi}{6} + \pi n, \quad n \in Z.$$

Now we can practice by solving some trigonometric inequalities.

Problem 108 Solve the inequality $|\sin x| > |\cos x|$.

Solution It is useful to know that any modulus inequality of the type

$$|f(x)| \geq |g(x)| \text{ or } |f(x)| \leq |g(x)|$$

can be rewritten in a different but equivalent form as

$f^2(x) \geq g^2(x)$ or $f^2(x) \leq g^2(x)$, respectively.

Therefore, we will solve

$$\begin{aligned}
&\sin^2 x > \cos^2 x \\
&\cos^2 x - \sin^2 x < 0 \\
&\cos 2x < 0 \\
&\frac{\pi}{2} + 2\pi n < 2x < \frac{3\pi}{2} + 2\pi n \\
&\frac{\pi}{4} + \pi n < x < \frac{3\pi}{4} + \pi n, \quad n \in Z
\end{aligned}$$

While solving this inequality, you could use Figure 3.14. Cosine of any argument is less than zero if the argument (angle) belongs to the interval

$\left(\frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n\right)$.

Dividing each term by 2 we obtain the answer.

Answer $\frac{\pi}{4} + \pi n < x < \frac{3\pi}{4} + \pi n$.

Problem 109 Solve the equation $\sqrt{-\cos x} = \sqrt{-1 + 2\sin^2 x}$.

Solution In order to solve this equation correctly, we need to make the quantities under the radicals nonnegative. The equation is equivalent to the following system:

$$\begin{cases} -\cos x = -1 + 2\sin^2 x \\ \cos x \le 0 \\ -1 + 2\sin^2 x \ge 0 \end{cases}$$

Using a main trigonometric identity we can rewrite the first equation as quadratic in $\cos x$. Now the system has the following form:

$$\begin{cases} 2\cos^2 x - \cos x - 1 = 0 \\ \cos x \le 0 \\ \cos 2x \le 0 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} \cos x = 1 \\ \cos x = -\dfrac{1}{2} \end{array} \right. \\ \cos x \le 0 \\ \cos 2x \le 0 \end{cases} \Leftrightarrow \cos x = -\frac{1}{2} \Leftrightarrow x = \pm\frac{2\pi}{3} + 2\pi n.$$

Remark While solving this problem we used the fact obtained in the previous problem. Remember that the solution of $\cos 2x \le 0$ is

$\frac{\pi}{4} + \pi n \le x \le \frac{3\pi}{4} + \pi n, \ n \in Z$.

If we place these restrictions on the unit circle, we can see that our solution $x = \pm\dfrac{2\pi}{3} + 2\pi n$ is inside of the given region (Figure 3.26).

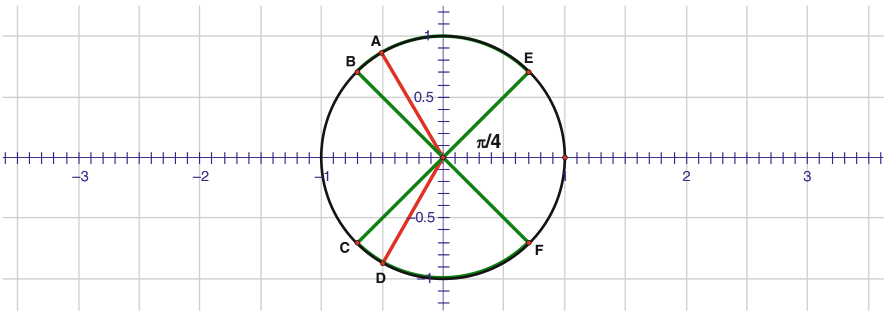


Figure 3.26 Sketch for Problem 109

3.3 Solving Miscellaneous Trigonometric Equations

Some additional basic trigonometric identities can be summarized as follows. We will use them in order to solve trigonometric problems:

$$
\begin{aligned}
&\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}\\
&\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}\\
&\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}\\
&\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}\\
&\cos\alpha\cdot\cos\beta = \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta))\\
&\sin\alpha\cdot\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))\\
&\sin\alpha\cdot\cos\beta = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))\\
&\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\\
&\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\\
&\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\\
&\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta\\
&\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\\
&\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}\\
&\sin 2\alpha = 2\sin\alpha\cos\alpha\\
&\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1\\
&\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}\\
&\sin 2\alpha = \frac{2\tan\alpha}{1 + \tan^2\alpha}\\
&\cos 2\alpha = \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha}\\
&\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha\\
&\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha
\end{aligned}
\tag{3.21}
$$

Below I will show the most popular and effective methods of solving trigonometric equations.

3.3.1 Factoring

If we can set up the equation so that the right side will be zero, and factor, then this will allow us to solve the more complicated equation as a set of simpler problems:

Example Solve $\sin x + \sin 5x = 0$.

Solution Rewriting the sum as a product using a corresponding formula from (3.21), we obtain

$$\sin x + \sin 5x = 0$$
$$2\sin 3x \cdot \cos 2x = 0$$

The solution to this example will be reduced to the solution of two much easier problems:

$$\sin 3x = 0 \qquad \cos 2x = 0, \text{ etc.}$$

Problem 110 $\cos 2x = 1 - \sin x$.

Solution

$$\begin{aligned}
&\cos 2x = 1 - \sin x\\
&1 - 2\sin^2 x = 1 - \sin x\\
&\sin x \cdot (2\sin x - 1) = 0\\
&\sin x = 0 \qquad \sin x = \frac{1}{2}\\
&x = \pi n \qquad\qquad x = (-1)^m \frac{\pi}{6} + \pi m, \quad n, m \in Z.
\end{aligned}$$

Problem 111 Solve $\cos 3x + \sin 2x - \sin 4x = 0$.

Solution We want to factor this equation and it is easy to rewrite the difference of two sines as a product using a formula from (3.21):

$$\cos 3x + (\sin 2x - \sin 4x) = 0$$
$$\cos 3x - 2\sin x \cos 3x = 0$$
$$\cos 3x(1 - 2\sin x) = 0$$
$$\left[\begin{array}{l}\cos 3x = 0 \Rightarrow \quad x = \dfrac{\pi}{6} + \dfrac{\pi}{3} \cdot n\\[2ex] \sin x = \dfrac{1}{2} \Rightarrow \quad x = (-1)^k \dfrac{\pi}{6} + \pi k, \quad n, k \in Z\end{array}\right.$$

If you put both solutions on the unit circle, you would see that the second solution is included in the first solution. Hence the answer can be written as

$$x = \frac{\pi}{6} + \frac{\pi}{3} \cdot n, \quad n \in Z.$$

Answer $x = \frac{\pi}{6} + \frac{\pi}{3} \cdot n, \quad n \in Z.$

3.3.2 Rewriting a Product as Sum or Difference

A factoring method works well only if the other side of an equation is zero or can become zero. Sometimes you need to do the opposite and rewrite a factorized expression as the sum or difference of trigonometric functions, so you can look for common factors. Let us solve the following problem.

Problem 112 Solve $\sin 5x \cos 3x = \sin 6x \cos 2x$.

Solution In order to find the solution to this equation we need to recall the following formula:

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta))$$

We obtain

$$\frac{1}{2}(\sin(8x) + \sin(2x)) = \frac{1}{2}(\sin(8x) + \sin(4x))$$
$$\sin 2x = \sin 4x$$

This simple equation must be solved by moving $\sin 4x$ to the left side and by applying the formula for the sine of a double angle (Method 1) or by applying the formula for the difference of two sines (Method 2). Please do not equate the arguments.

Method 1.

$$\sin 2x - 2\sin 2x \cos 2x = 0$$
$$\sin 2x(1 - 2\cos 2x) = 0$$
$$\sin 2x = 0 \qquad \cos 2x = \frac{1}{2}$$
$$2x = \pi n \qquad 2x = \pm\frac{\pi}{3} + 2\pi k$$
$$x = \frac{\pi n}{2} \quad \text{or} \quad x = \pm\frac{\pi}{6} + \pi k$$

Method 2.

$$\sin 2x - \sin 4x = 0$$
$$-2\sin x \cos 3x = 0$$
$$\sin x = 0 \quad \text{or} \quad \cos 3x = 0$$
$$x = \pi n \quad or \quad 3x = \frac{\pi}{2} + \pi k \Leftrightarrow x = \frac{\pi}{6} + \frac{\pi k}{3}$$

If you place both solutions on the unit circle, then you will see that the answers obtained in different ways are identical.

3.3.3 Reducing The Degree of Trigonometric Functions

From the identities above, we know that squares of sine or cosine can be replaced by a linear expression in a double angle. For example, let us solve the following:

Problem 113 Solve $\sin^2 x + \sin^2 3x = 1$.

Solution Rewriting the sine of single angles in terms of the cosine of a corresponding double angle we obtain

$$\begin{aligned}&\sin^2 x + \sin^2 3x = 1\\&\frac{1-\cos 2x}{2} + \frac{1-\cos 6x}{2} = 1\\&\cos 2x + \cos 6x = 0\end{aligned}$$

Because the right side is zero, we can factor the left side as

$$\cos 2x + \cos 6x = 0$$

$$2\cos 4x \cdot \cos 2x = 0$$

We will proceed as follows:

$$\cos 4x = 0, \quad \cos 2x = 0, \text{ etc.}$$

Answer $\left[\begin{array}{l} x = \dfrac{\pi}{8} + \dfrac{\pi n}{4} \\ x = \dfrac{\pi}{4} + \dfrac{\pi k}{2}, \quad n, k \in Z \end{array}\right.$.

3.3.4 Homogeneous Trigonometric Equations

An equation $a_n \sin^n x + a_{n-1} \sin^{n-1} x \cos x + \ldots + a_1 \sin x \cos^{n-1} x = 0$ is called a homogeneous trigonometric equation of nth order. Such equations will be reduced to a polynomial type equation by dividing either by sine or cosine raised to the highest degree.

Problem 114 Solve the equation $3\sin^2 x - 3\sin x\cos x + 4\cos^2 x = 0$.

Solution Let us divide both sides by $\cos^2 x \neq 0$:

$$3\tan^2 x - 3\tan x + 4 = 0$$
$$D = 9 - 48 = -39 < 0.$$

There are no real solutions.

Let us change this problem a little bit and make the right side 2. The following problem will have a solution.

Problem 115 Solve the equation $3\sin^2 x - 3\sin x\cos x + 4\cos^2 x = 2.$

Solution Let us replace the right side by $2\cos^2 x + 2\sin^2 x = 2$, collect like terms, and obtain an equivalent equation:

$$\sin^2 x - 3\sin x\cos x + 2\cos^2 x = 0$$

We will again divide both sides by $\cos^2 x \neq 0$:

$$\tan^2 x - 3\tan x + 2 = 0$$
$$\tan x = 1 \quad \text{or} \quad \tan x = 2$$
$$x = \frac{\pi}{4} + \pi n \quad \text{or} \quad x = \text{arctan}2 + \pi k, \quad n, k \in Z.$$

A simple example of a homogeneous equation is $a\sin x + b\cos x = 0$ that is a homogeneous trigonometric equation of the first order.

Dividing both sides by $\cos x \neq 0$ (or $\sin x \neq 0$) we obtain

$$\tan x = -\tfrac{b}{a}, \text{ etc.}$$

However, if we change the right side of the original equation, and make it $a\sin x + b\cos x = c$, then dividing by sine or cosine would not help and I recommend the method of using an auxiliary argument (angle), as explained below.

A homogeneous trigonometric equation of the first order can be written as

$$A\sin x \pm B\cos x = 0, \quad A^2 + B^2 \neq 0 \tag{3.22}$$

Such equations can be solved using two different methods:

Method 1: Assuming that either $\sin x \neq 0$, $\cos x \neq 0$ we can divide (3.22) either by **sin *x*** or **cos *x*** and obtain a new equation in cot x or tan x, respectively.

For example:

$$A\tan x = -B, \quad \tan x = -\tfrac{B}{A}, \text{ etc.}$$

Method 2: Recalling the following trigonometric identities

$$\begin{aligned} \sin(x+y) &= \sin x\cos y + \cos x\sin y \\ \sin(x-y) &= \sin x\cos y - \cos x\sin y \\ \cos(x+y) &= \cos x\cos y - \sin x\sin y \\ \cos(x-y) &= \cos x\cos y + \sin x\sin y \end{aligned}$$

we can divide and multiply (3.22) by $\sqrt{A^2+B^2}$:

$$A\sin x \pm B\cos x = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\sin x \pm \frac{B}{\sqrt{A^2+B^2}}\cos x\right)$$

Next, we will introduce ***an auxiliary angle***:

$$\varphi = \arcsin\frac{B}{\sqrt{A^2+B^2}};\ \varphi = \arctan\frac{B}{A},\quad A \neq 0 \tag{3.23}$$

Therefore, (3.22) can be rewritten as

$$\begin{aligned} A\sin x \pm B\cos x &= \sqrt{A^2+B^2}\cos\varphi\sin x \pm \sin\varphi\cos x \\ &= \sqrt{A^2+B^2}\sin(x\pm\varphi) \end{aligned} \tag{3.24}$$

Formula (3.24) is rarely introduced in the high school curriculum. However, this formula allows us to easily find a maximum or minimum of the following function $f(x) = A\sin x + B\cos x$ and it is absolutely necessary when solving equations like this: $A\sin x + B\cos x = C$.

Remark If for $A\sin x + B\cos x,\ \ B \geq 0$, then you choose $\varphi = \arcsin\frac{B}{\sqrt{A^2+B^2}}$.

If $B < 0$, then you choose $\varphi = -\arccos\frac{A}{\sqrt{A^2+B^2}}$.

Let us practice this formula by solving two problems below.

Problem 116 Solve the equation $3\sin x - 4\cos x = 5$.

Solution Since $\sqrt{3^2+4^2} = 5$, then the equation can be rewritten as

$$\begin{aligned} &\sin(x-\varphi) = 1, \quad \varphi = \arccos\left(\frac{3}{5}\right) \\ &x - \varphi = \frac{\pi}{2} + 2\pi n \\ &x = \arccos\left(\frac{3}{5}\right) + \frac{\pi}{2} + 2\pi n, \quad n \in Z \end{aligned}$$

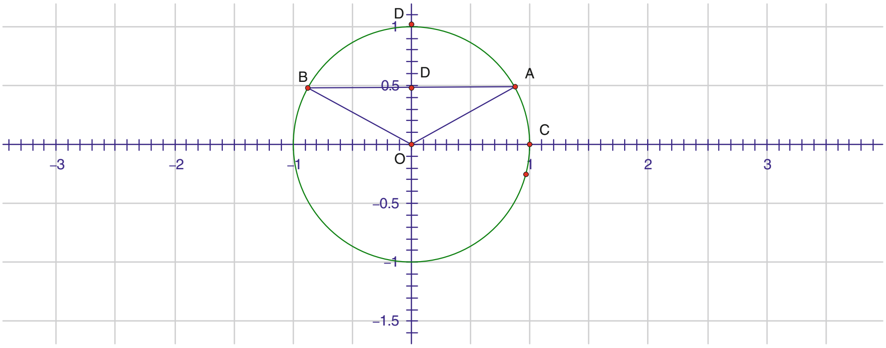


Figure 3.27 Sketch for Problem 117

Problem 117 Find solutions to $4\sin 2x + 3\cos 2x = 5$ for all $x \in [0, \pi]$.

Solution Using an ***auxiliary angle*** formula (3.23), we can rewrite the equation as follows:

$$
\begin{aligned}
&4\sin 2x + 3\cos 2x = 5\\
&5\sin(2x+\varphi) = 5, \varphi = \arcsin\frac{3}{5}\\
&\sin(2x+\varphi) = 1
\end{aligned}
$$

As we know there is only one point on the unit circle within each revolution that has the second coordinate 1. In the graph above it is point D. Point D corresponds to the angle $\frac{\pi}{2} + 2\pi n, n = 0, \pm 1, \pm 2, \ldots$ (Figure 3.27).

Therefore

$$
\begin{aligned}
2x + \varphi &= \frac{\pi}{2} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots\\
x &= \frac{\pi}{4} - \frac{1}{2}\cdot \arcsin\frac{3}{5} + \pi n, \quad n \in Z.
\end{aligned}
$$

Since we need to find only solutions in the first and the second quadrants, then $n = 0$, and $x = \frac{\pi}{4} - \frac{1}{2}\cdot\arcsin\frac{3}{5}$.

Answer $x = \frac{\pi}{4} - \frac{1}{2}\cdot\arcsin\frac{3}{5}$.

Problem 118 Solve $\sin x + \cos x = -1$.

Solution This problem can be attacked in several ways; some of them could lead to the wrong answer, if proper restrictions are not applied.

Method 1: Using an auxiliary angle the given equation can be written as

$$\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)=-1$$

The solution is obvious

$$\sin\left(x+\tfrac{\pi}{4}\right)=\tfrac{-1}{\sqrt{2}}, \text{ etc.}$$

Method 2: Using a double-angle formula, we can write $\sin x$ and $\cos x$ in terms of the tangent of a half angle. Note that the formula is proved in Section 3.4.2:

$$\begin{aligned}
&\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=-1\\
&\tan\frac{x}{2}=-1\\
&x=-\frac{\pi}{2}+2\pi n
\end{aligned}$$

Method 3 (maybe not the best).
Some students try to square both sides of the equation and obtain

$$\sin x\cdot\cos x=0$$
$$\left[\begin{array}{l}\sin x=0\Rightarrow x=\pi n\\ \cos x=0\Rightarrow x=\dfrac{\pi}{2}+\pi k\end{array}\right.$$

This solution is wrong because in the original equation the right side was negative, and when we squared both sides we replaced our equation by an equation that is not equivalent.

Question What restrictions can be added to the solution of the problem so we would still get the correct answer?

Problem 119 It is known that function $f(x)=A\cos x+B\sin x$ has zeros at two different values of x such that $f(x_1)=f(x_2)=0$. Prove that $x_1-x_2=n\pi, n\in Z$.

Proof Let us rewrite the function as

$$f(x) = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos x + \frac{B}{\sqrt{A^2+B^2}}\sin x\right) = \sqrt{A^2+B^2}\sin(x+\varphi),$$
$$\sin\varphi = \frac{A}{\sqrt{A^2+B^2}}, \quad \cos\varphi = \frac{B}{\sqrt{A^2+B^2}}$$

Next, using the condition of the problem we can state that

$$\sin(x_1+\varphi) = \sin(x_2+\varphi)$$

Answer $x_1 - x_2 = \pi n,\ n \in Z.$

3.3.5 Selecting Root Subject to Conditions

Sometimes, after solving an equation, we need to select only such roots that satisfy a given condition.

Problem 120 Solve $\log_{\sin(-x)}\left(\sin\frac{x}{2} + \sin\frac{3x}{2}\right) = 1.$

Solution In order to solve this equation we have to use properties of logarithms:

$$\begin{aligned} &\log_a b = c \quad \Leftrightarrow \quad a^c = b \\ &\textit{if} \ \ a > 0, \quad a \neq 1, \ \ b > 0 \end{aligned} \tag{3.25}$$

The following must be true:

$$\begin{cases} \sin(-x) > 0 \\ \sin(-x) \neq 1 \\ \sin(-x) = \sin\frac{x}{2} + \sin\frac{3x}{2} \end{cases}$$

Sine is an odd function, and the right side of the last equation can be written as a product:

$$\begin{cases} \sin(-x) > 0 \\ \sin(-x) \neq 1 \\ -\sin x = 2\sin x \cdot \cos\frac{x}{2} \end{cases}$$

After factoring $\sin x$ in the last equation and using the fact that $\sin x$ must be less than zero, we have

$$\begin{cases} \sin x < 0 \\ \sin x \neq -1 \\ \cos \frac{x}{2} = -\frac{1}{2} \end{cases}$$

The last equation has solution $x = \pm\frac{4\pi}{3} + 4\pi n, n \in Z$. However, only $x = \frac{4\pi}{3} + 4\pi n$, $n \in Z$ is in the region where $\sin x < 0$.

Answer $x = \frac{4\pi}{3} + 4\pi n, n \in Z$.

3.3.6 Completing a Square

Completing a square often helps to reduce the order of an equation and dramatically simplify it as well. Let use learn this by solving the problem below.

Problem 121 (Chirsky) Solve the equation $\sin^6 x + \cos^6 x = \frac{3}{4}(\cos^4 x + \sin^4 x)$.

Solution The idea is to complete the square on both sides and replace $\sin^2 x + \cos^2 x = 1$.

$$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x = 1 - \frac{1}{2}\sin^2 2x = \frac{3}{4} + \frac{1}{4}\cos 4x$$

$$\sin^6 x + \cos^6 x = (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x) = \frac{5}{8} + \frac{3}{8}\cos 4x.$$

After simplification, our equation becomes

$$\frac{5}{8} + \frac{3}{8}\cos 4x = \frac{3}{4}\left(\frac{3}{4} + \frac{1}{4}\cos 4x\right)$$

$$\cos 4x = -\frac{1}{3}$$

$$x = \pm\arccos\frac{1}{3} + \frac{\pi n}{2},\ n \in Z$$

Answer $x = \pm\frac{1}{4}\arccos\frac{1}{3} + \frac{\pi}{2}\cdot n,\ \ n \in Z.$

3.4 Proofs of Some Trigonometric Identities

When I was a high school student, I chose to memorize these formulas; I still remember all of them by heart. The younger generation is now able to find things immediately on the Internet, and does not tend to memorize as we did.

I disagree with this practice and am surprised that even some math teachers never remember formulas. If you want to solve complex problems, then remembering as many formulas as you can is helpful. When you see a problem, you can quickly "play" different scenarios in your mind based on your experience and knowledge of formulas. If you never tried to memorize any formulas, then you would not see many "easy" solutions that come immediately to mind. In this section, we show some proofs for important trigonometric formulas using the unit circle and basic knowledge of plane geometry.

3.4.1 Angle Addition Formulas

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{aligned} \tag{3.26}$$

Consider Point C (1,0) on the unit circle. Let $\angle COA = \alpha$ and $\angle AOB = \beta$ (Figure 3.28).

In the original, "blue" coordinate system, the coordinates of point C and point B are $(1, 0)$ and $(\cos(\alpha+\beta), \sin(\alpha+\beta))$, respectfully. Since the distance between two points is preserved despite a coordinate system, we will do the following.

1. Evaluate the square of the distance BC as

$$BC^2 = \sin^2(\alpha+\beta) + (\cos(\alpha+\beta) - 1)^2 = -2\cos(\alpha+\beta) + 2.$$

2. Draw a new coordinate system such that X goes through side OA and the Y-axis is perpendicular to OA. In this coordinate system point C has coordinates C

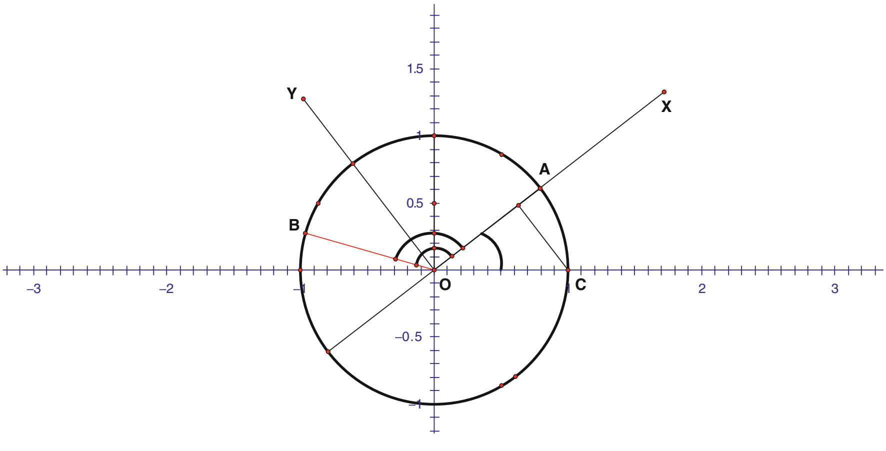


Figure 3.28 Angle addition formulas

$(\cos\alpha, \sin(-\alpha)) = (\cos\alpha, -\sin\alpha)$ and point B has simply coordinates $(\cos\beta, \sin\beta)$. Thus the square of the distance between C and B is

$$\begin{aligned} BC^2 &= (\sin\beta + \sin\alpha)^2 + (\cos\beta - \cos\alpha)^2 \\ &= 2 + 2\sin\alpha\sin\beta\text{-}2\cos\alpha\cos\beta. \end{aligned}$$

Equating the right sides of the square of the distance equations, we obtain

$$-2\cos(\alpha+\beta) + 2 = 2 + 2\sin\alpha\sin\beta - 2\cos\alpha\cos\beta$$

From which we immediately obtain the first formula in (3.26):

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

The proof is completed.

The formula for sine of the sum of two angles can be easily proven if we use the complementary angle property (3.4):

$$\begin{aligned} \sin(\alpha+\beta) &= \cos\left(\frac{\pi}{2} - (\alpha+\beta)\right) = \cos\left((\alpha+\beta) - \frac{\pi}{2}\right) = \cos\left(\alpha + \left(\beta - \frac{\pi}{2}\right)\right) \\ &= \cos\alpha\cos\left(\beta - \frac{\pi}{2}\right) - \sin\alpha\sin\left(\beta - \frac{\pi}{2}\right) \\ &= \cos\alpha\sin\beta + \sin\alpha\cos\beta \end{aligned}$$

This proves the first addition formula in (3.26):

From (3.26) we can always derive the so-called double-angle formulas.

Denoting $\beta = \alpha$, we obtain

$$\begin{aligned} \sin 2\alpha &= 2\sin\alpha\cos\alpha \\ \cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \end{aligned} \tag{3.27}$$

The last formula can also be rewritten in two other convenient forms below, either in terms of $\cos^2\alpha$ or $\sin^2\alpha$ by using a Pythagorean identity:

$$\cos(2\alpha) = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \tag{3.28}$$

Problem 122 Evaluate without a calculator $\cos\dfrac{\pi}{12}$.

Solution

Method 1: Since $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$, then

$$\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1+\sqrt{3}}{2\sqrt{2}}$$

Method 2: Since we know that $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, we can apply the formula of the cosine of a double angle as

$$2\cos^2\frac{\pi}{12} = 1 + \cos\frac{\pi}{6}$$

$$\cos\frac{\pi}{12} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\left(\sqrt{3}+1\right)^2}{2\cdot 4}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Answer $\dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

Remark In Method 2, we had to demonstrate our abilities to work with radicals. Thus $\sqrt{3}+2$ can be written as

$$2+\sqrt{3} = \frac{\left(1+\sqrt{3}\right)^2}{2} = \frac{1+2\sqrt{3}+3}{2} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3}.$$

3.4.2 Double-Angle Formulas

Many formulas can be easily proven geometrically on the unit circle.

Using Figure 3.29, let us prove the following formulas:

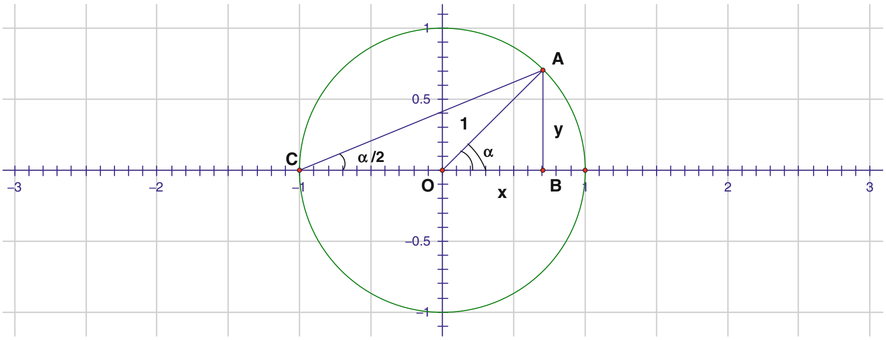


Figure 3.29 Proof of double-angle formulas

$$
\begin{aligned}
\sin\alpha &= \frac{2\tan\dfrac{\alpha}{2}}{1+\tan^2\dfrac{\alpha}{2}} \\
\cos\alpha &= \frac{1-\tan^2\dfrac{\alpha}{2}}{1+\tan^2\dfrac{\alpha}{2}} \\
\tan\alpha &= \frac{2\tan\dfrac{\alpha}{2}}{1-\tan^2\dfrac{\alpha}{2}}
\end{aligned}
\tag{3.29}
$$

Let us take a point A with coordinates (x, y) on the unit circle with center O. Since the point is associated with angle α, then $x = OB = \cos\alpha, y = AB = \sin\alpha$. Drop perpendicular line AB to the X-axis and then connect point A with point C on the diameter. From Euclidean geometry we know that if $\angle AOB = \alpha$, then $\angle ACB = \frac{\alpha}{2}$ as central and inscribed angles, respectfully.

From triangle ABC we have $\tan\frac{\alpha}{2} = \frac{AB}{CB} = \frac{y}{1+x}$ from which we obtain

$$y = \tan\frac{\alpha}{2}\cdot(1+x) \tag{3.30}$$

On the other hand, from triangle ABO we obtain that $x^2 + y^2 = 1$ or

$$y^2 = 1 - x^2 \tag{3.31}$$

Squaring both sides of (3.30) and applying the difference of squares formula to (3.31)

$$\tan^2\frac{\alpha}{2}\cdot(1+x)^2 = (1-x)(1+x)$$

Dividing both sides by $(1+x)$ we obtain

$$\tan^2\frac{\alpha}{2} + x\cdot\tan^2\frac{\alpha}{2} = 1 - x$$

Factoring terms with x:

$$x\left(1+\tan^2\frac{\alpha}{2}\right) = 1 - \tan^2\frac{\alpha}{2}$$

we obtain the required formula:

$$x = \cos\alpha = \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \tag{3.32}$$

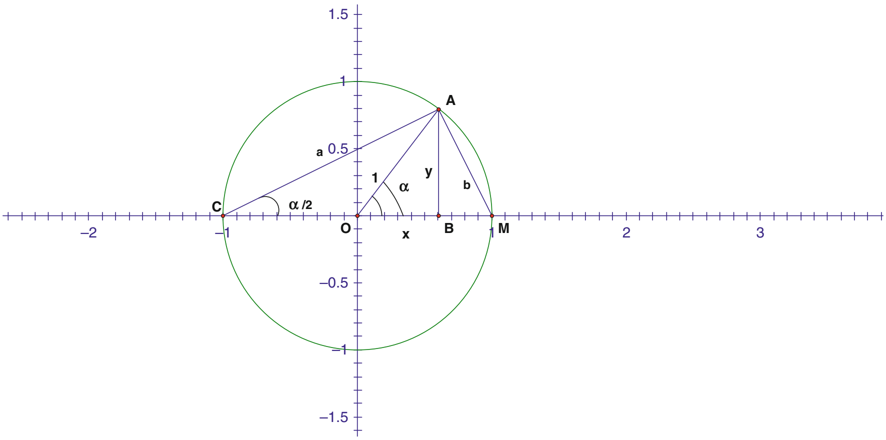


Figure 3.30 Geometric proof of trigonometric formulas

The formula for y immediately follows from (3.30) or (3.31). Please obtain it on your own:

$$y = \sin\alpha = \frac{2\tan\frac{\alpha}{2}}{1 + \tan^2\frac{\alpha}{2}} \tag{3.33}$$

Additionally, dividing (3.33) by (3.32) we obtain the last formula for (3.29):

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1 - \tan^2\frac{\alpha}{2}} \tag{3.34}$$

Let us now prove formulas for the sine and cosine of a double angle. We will modify the previous picture a little bit by connecting point A with M of the diameter (Figure 3.30). From geometry we know that MAC is the right triangle, and MAC is a right angle because MC is a diameter. Let $AC = a$ and $AM = b$. Express the area of triangle MAC in two different ways, first, using the half product of the legs and second, the half product of the height AB and base CM:

$$a \cdot b = 2 \cdot y \tag{3.35}$$

From triangle ABC we have

$$y = a\sin\frac{\alpha}{2} \tag{3.36}$$

Since triangles ABM and ABC are similar ($\Delta ABM \sim \Delta ABC$), the corresponding angles are equal, and then from ΔABM we have

$$y = b \cos \frac{\alpha}{2} \tag{3.37}$$

Multiplying the left and right sides of (3.36) and (3.37) we obtain that

$$ab = \frac{y^2}{\sin \frac{\alpha}{2} \cdot \cos \frac{\alpha}{2}} \tag{3.38}$$

Equating the right sides of (3.35) and (3.38) and after simplification we will have that

$$y = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$$

Replacing $y = \sin \alpha$ we prove the formula

$$\sin \alpha = 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}$$

Next, let us prove that $\cos \alpha = 2 \cos^2 \frac{\alpha}{2} - 1$. We can again use Figure 3.30.

From the right triangle OAB we see that $|OB| = x = \cos \alpha$.

Applying the Pythagorean Theorem to the right triangle BAM, we obtain

$$\begin{aligned}
&|BM|^2 + |AB|^2 = |AM|^2 \\
&(1 - x)^2 + y^2 = b^2 \\
&1 - 2x + (x^2 + y^2) = b^2 \\
&2 - 2x = b^2
\end{aligned}$$

From this we can obtain

$$b^2 = 2(1 - x) \tag{3.39}$$

On the other hand, substituting (3.37) into (3.39) we obtain a new formula:

$$\frac{y^2}{\cos^2 \frac{\alpha}{2}} = 2(1 - x)$$

From the right triangle OBA we have $x^2 + y^2 = 1$ and also $x = \cos \alpha$, and finalize our proof:

$$\begin{aligned}
&\frac{1 - x^2}{\cos^2 \frac{\alpha}{2}} = 2(1 - x) \\
&1 + x = 2 \cos^2 \frac{\alpha}{2} \\
&x = 2 \cos^2 \frac{\alpha}{2} - 1 \\
&\cos \alpha = 2 \cos^2 \frac{\alpha}{2} - 1
\end{aligned}$$

In order to memorize these formulas, I want you to practice solving the following two problems.

Problem 123 Solve $\sin x + \cos x = 1$.

Solution Let use replace both function in terms of tangent of half angle ((3.32) and (3.33)):

$$\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = 1$$

$$2\tan\frac{x}{2} - 2\tan^2\frac{x}{2} = 0$$

$$\tan\frac{x}{2} - 2\tan^2\frac{x}{2} = 0$$

$$1.\ \tan\frac{x}{2} = 0 \quad 2.\ \tan\frac{x}{2} = 1$$

$$\frac{x}{2} = \pi n \quad \text{or} \quad \frac{x}{2} = \frac{\pi}{4} + \pi k$$

$$x = 2\pi n \quad \text{or} \quad x = \frac{\pi}{2} + 2\pi k, \quad n, k \in Z$$

Answer $x = 2\pi n; x = \frac{\pi}{2} + 2\pi k, \ \ n, k \in Z.$

The following problem would require a similar substitution.

Problem 124 Find all solutions to the equation $\left(1 - \tan\frac{x}{2}\right) \cdot (1 + \sin x) = 1 + \tan\frac{x}{2} - \cos x.$

Solution The idea is to rewrite the entire equation in terms of one function, for example, tangent of half angle: $\left(1 - \tan\frac{x}{2}\right)\left(1 + \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) = 1 + \tan\frac{x}{2} - \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}.$

Denote

$$y = \tan\frac{x}{2} \tag{3.40}$$

and after substitution and simplification, we obtain the following polynomial equation:

$$(1-y)(y+1)^2 = y(y+1)^2$$

$$(y+1)^2(1-2y) = 0$$

$$1.\ \ y = -1 \quad \text{or} \quad 2.\ \ y = \frac{1}{2}$$

Next, we will solve (3.40) for each value of variable y:

1. $$\tan\frac{x}{2} = -1$$
$$x = -\frac{\pi}{2} + 2\pi k$$

2. $$\tan\frac{x}{2} = \frac{1}{2}$$
$$x = 2\arctan\frac{1}{2} + 2\pi n$$

Answer $-\frac{\pi}{2} + 2\pi k;\ \ 2\arctan\frac{1}{2} + 2\pi n, \quad n, k \in Z.$

3.4.3 Triple Angles and More: Euler–De Moivre's Formulas

I personally never memorize formulas for the sine or cosine of a triple angle, or formulas for $\cos 4\alpha$ or $\sin 5\alpha$, etc. I believed that they can be derived each time you need them.

I am sharing my experience and do not want to make you repeat exactly what I did or did not do. However, I learned all these ideas and created my own methods by reading many books, and, more importantly, by practicing the solution of a lot of problems.

Let us see how the sine or cosine of a triple angle can be derived from (3.26)–(3.28):

$$\begin{aligned}\sin(2\alpha + \alpha) &= \sin 2\alpha \cos\alpha + \cos 2\alpha \sin\alpha \\ &= 2\sin\alpha\cos^2\alpha + (1 - 2\sin^2\alpha)\sin\alpha \\ &= 2\sin\alpha(1 - \sin^2\alpha) + (1 - 2\sin^2\alpha)\sin\alpha \\ &= 3\sin\alpha - 4\sin^3\alpha\end{aligned}$$

Using the same approach, please derive $\cos 3\alpha$ in terms of the cosine of a single angle on your own. Together, we have

$$\begin{aligned}\sin 3\alpha &= 3\sin\alpha - 4\sin^3\alpha \\ \cos 3\alpha &= 4cos^3\alpha - 3\cos\alpha\end{aligned} \tag{3.41}$$

Remark These formulas can be proven by using complex numbers, Euler and De Moivre's Theorem:

$$\left.\begin{aligned} e^{i\varphi} &= \cos\varphi + i\sin\varphi \\ e^{i\varphi\cdot n} &= \cos\varphi n + i\sin\varphi n\end{aligned}\right\} \Rightarrow (\cos\varphi + i\sin\varphi)^n = \cos\varphi n + i\sin\varphi n$$

In order to prove the double-angle formula we will use $n = 2$:

$$(\cos\varphi + i\sin\varphi)^2 = \cos 2\varphi + i\sin 2\varphi$$

The idea is to raise the left side to the corresponding power and then equate real and imaginary parts. We also need to use the property that $i^2 = -1$.

$$\cos^2\varphi + i\cdot 2\sin\varphi\cos\varphi - \sin^2\varphi = \cos 2\varphi + i\sin 2\varphi$$
$$(\cos^2\varphi - \sin^2\varphi) + i\cdot 2\sin\varphi\cos\varphi = \cos 2\varphi + i\sin 2\varphi$$

Equating real and imaginary parts of both sides, we get double-angle formulas:

$$\cos^2\varphi - \sin^2\varphi = \cos 2\varphi$$
$$2\sin\varphi\cos\varphi = \sin 2\varphi$$

Similarly using cube of a sum and the formula below, we will get the formulas for triple angles:

$$(\cos\varphi + i\sin\varphi)^3 = \cos 3\varphi + i\sin 3\varphi$$

\- -

Because these formulas are not in the high school curriculum, we will demonstrate how they can be obtained from each other and used for proving some trigonometric identities. Euler stated the following formula:

$$e^{ix} = \cos x + i\sin x \tag{3.42}$$

On one hand, replacing x by nx in (3.42) we obtain De Moivre's formula:

$$e^{inx} = \cos nx + i\sin nx$$

On the other hand, raising both sides of (3.42) to the second, third, fourth, and nth power, we must get the following true chain of the equations:

$$\begin{aligned}
(e^{ix})^2 &= (\cos x + i\sin x)^2 \\
e^{i2x} &= \cos 2x + i\sin 2x \\
(e^{ix})^3 &= (\cos x + i\sin x)^3 \\
e^{i3x} &= \cos 3x + i\sin 3x \\
(e^{ix})^4 &= (\cos x + i\sin x)^4 \\
e^{i4x} &= \cos 4x + i\sin 4x \\
&- - - - - - - - - - - - - \\
e^{inx} &= \cos nx + i\sin nx
\end{aligned}$$

Using the fact that $\forall n,\ e^{inx} = \cos nx + i\sin nx = (\cos x + i\sin x)^n$, we can easily derive many well-known formulas for the cosine or sine of double, triple, etc. angles.

For example, let us prove the formulas for sine and cosine of a triple angle:

$$\cos 3x + i\sin 3x = (\cos x + i\sin x)^3$$
$$\cos 3x + i\sin 3x = \cos^3 x + 3\cos^2 x i\sin x + 3\cos x\cdot i^2\sin^2 x + i^3\sin^3 x$$
$$\cos 3x + i\sin 3x = \cos^3 x + i\cdot 3\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x$$
$$\cos 3x + i\sin 3x = (\cos^3 x - 3\cos x\sin^2 x) + i(3\cos^2 x\sin x - \sin^3 x)$$

By equating terms with and without i and using a Pythagorean identity, we obtain

$$\begin{aligned}&\cos 3x = \cos^3 x - 3\cos x\sin^2 x = 4\cos^3 x - 3\cos x\\ &\sin 3x = 3\sin x - 4\sin^3 x\end{aligned}$$

We can also derive any formulas for sine or cosine of the sum of two angles or their difference.

Thus, consider Euler's formula (3.42) for two different arguments, x, y.

Next, we will multiply the left and right sides:

$$e^{ix}\cdot e^{iy} = (\cos x + i\sin x)\ (\cos y + i\sin y)$$

Simplifying left and right sides we will obtain

$$e^{i(x+y)} = (\cos x\cos y - \sin x\sin y) + i(\sin x\cos y + \sin y\cos x) \tag{3.43}$$

Because the left side of formula (3.43) can also be written as

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y) \tag{3.44}$$

From (3.43) and (3.44) we obtain other very familiar formulas:

$$\begin{aligned}\cos(x+y) &= \cos x\cos y - \sin x\sin y\\ \sin(x+y) &= \sin x\cos y + \cos x\sin y\end{aligned} \tag{3.45}$$

Equation (3.45) can easily be changed to the formulas for cosine and sine of the difference. We need to remember that cosine is even and sine is an odd function. Thus we have

$$\begin{aligned}\cos(x-y) &= \cos x\cos y + \sin x\sin y\\ \sin(x-y) &= \sin x\cos y - \cos x\sin y\end{aligned} \tag{3.46}$$

Now, if we add the first formulas of (3.45) and (3.46), we will obtain another well-known formula for the product of two cosine functions:

$$\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y)) \tag{3.47}$$

By subtracting the first formula of (3.46) and the first formula of (3.45), we will obtain the product of two sines:

$$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y)) \tag{3.48}$$

And, if we add the left and right sides of the second formulas of (3.45) and (3.46), we will get

$$\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y)) \tag{3.49}$$

Finally, from (3.48), and after substitution

$$\begin{cases} \alpha = x - y \\ \beta = x + y \end{cases} \Leftrightarrow \quad x = \frac{\alpha+\beta}{2}, \quad y = \frac{\beta-\alpha}{2},$$

we can obtain the formulas for the difference of cosine functions:

$$\cos\alpha - \cos\beta = 2\sin\frac{\alpha+\beta}{2} \cdot \sin\frac{\beta-\alpha}{2} \tag{3.50}$$

If you remember, we proved these formulas in a different way earlier in this chapter.

Problem 125 Evaluate $\sin 18°$ precisely.

Solution I remember when I was in 9th grade, a similar problem appeared at the city math Olympiad. I noticed that $\cos 36° = \sin 54°$ (this is based on the properties of complementary angles, e.g., $\cos\beta = \sin(90° - \beta)$), so $\sin 18°$ can be found by solving the equation

$$\sin 3\alpha = \cos 2\alpha, \ \ \alpha = 18°$$

I first rewrote $\sin 3\alpha$ as the sine of the sum of two angles and replaced the right-hand side in terms of the sine of a single angle:

$$\begin{aligned}
&\sin(2\alpha + \alpha) = 1 - 2\sin^2\alpha \\
&\sin\alpha\cos 2\alpha + \cos\alpha\sin 2\alpha = 1 - 2\sin^2\alpha \\
&\sin\alpha(1 - 2\sin^2\alpha) + \cos\alpha 2\sin\alpha\cos\alpha = 1 - 2\sin^2\alpha \\
&4\sin^3\alpha - 2\sin^2\alpha - 3\sin\alpha + 1 = 0
\end{aligned}$$

This equation can be factored as

$$(\sin\alpha - 1)\left(4\sin^2\alpha + 2\sin\alpha - 1\right) = 0.$$

We are not interested in $\sin\alpha = 1(\alpha = 90^\circ)$. Our answer comes from solving the quadratic equation $4\sin^2\alpha + 2\sin\alpha - 1 = 0$ and selecting the root that is less than 1:

$$\sin\alpha = \sin 18^\circ = \frac{\sqrt{5}-1}{4}.$$

Answer $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$.

Remark In my book "Methods of Solving Complex Geometry Problems" I used this approach in order to construct an angle of 36° by first constructing 18° and then by doubling it.

Problem 126 Evaluate $A = \sin 20^\circ \cdot \sin 40^\circ \cdot \sin 80^\circ$.

Solution Using formula (3.9) we will rewrite the product of the two first sines as the difference of cosines:

$$\begin{aligned} A &= \sin 20^\circ \cdot \sin 40^\circ \cdot \sin 80^\circ \\ &= \frac{1}{2}(\cos 20^\circ - \cos 60^\circ)\sin 80^\circ \\ &= \frac{1}{2}\cos 20^\circ \sin 80^\circ - \frac{1}{2}\cos 60^\circ \sin 80^\circ \\ &= \frac{1}{2}\cdot\frac{1}{2}(\sin 100^\circ + \sin 60^\circ) - \frac{1}{4}\cdot \sin 80^\circ \end{aligned}$$

Because $\sin 100^\circ = \sin 80^\circ$ as sines of supplementary angles, after simplification we obtain

$$A = \frac{1}{4}\sin 60^\circ = \frac{\sqrt{3}}{8}.$$

Answer $\frac{\sqrt{3}}{8}$.

Problem 127 Prove that $A = \cos\frac{\pi}{7}\cdot \cos\frac{4\pi}{7}\cdot \cos\frac{5\pi}{7} = \frac{1}{8}$.

Proof Note that $\cos(\pi - \alpha) = -\cos\alpha$; then $\cos\frac{5\pi}{7} = -\cos\left(\pi - \frac{5\pi}{7}\right) = -\cos\frac{2\pi}{7}$.
Using this we can rewrite the original expression as

$$A = -\cos\frac{\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{4\pi}{7}$$

Next, we will multiply it and divide by $2\sin\frac{\pi}{7}$. Applying the formula for the sine of a double angle twice we have

$$A = -\frac{\sin\frac{8\pi}{7}}{8\sin\frac{\pi}{7}} = -\frac{\sin\left(\pi+\frac{\pi}{7}\right)}{8\sin\frac{\pi}{7}} = \frac{\sin\frac{\pi}{7}}{8\sin\frac{\pi}{7}} = \frac{1}{8}.$$

The proof is completed.

3.5 Trigonometric Series

In this section we consider trigonometric series. In order to evaluate trigonometric series we need to know basic trigonometric identities, Euler and De Moivre's formulas. Next, we see how these formulas can be used for evaluating trigonometric series.

Problem 128 Prove that

$$S_n = \sin x + \sin 2x + \sin 3x + \ldots + \sin nx = \frac{\sin\frac{nx}{2}\cdot\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}.$$

Proof Multiplying the sum by $2\sin\frac{x}{2}$, and then using formula (3.48) for the product of two sine functions, we obtain the following:

$$\begin{aligned}
S_n\cdot 2\sin\frac{x}{2} &= 2\cdot\sin\frac{x}{2}\cdot(\sin x + \sin 2x + \sin 3x + \ldots + \sin nx)\\
&= 2\sin\frac{x}{2}\sin x + 2\sin\frac{x}{2}\sin 2x + \ldots + 2\sin\frac{x}{2}\sin nx\\
&= \cos\left(\frac{x}{2}-x\right) - \cos\left(\frac{x}{2}+x\right) + \cos\left(\frac{x}{2}-2x\right) - \cos\left(\frac{x}{2}+2x\right) +\\
&\quad \ldots + \cos\left(\frac{x}{2}-nx\right) - \cos\left(\frac{x}{2}+nx\right)
\end{aligned}$$

Because cosine is an even function, then $\cos(-y) = \cos y$ and all terms in the middle of the sum will be eliminated as follows:

$$\begin{aligned}
2\sin\frac{x}{2}\cdot S_n &= \cos\frac{x}{2} - \cos\frac{3x}{2} + \cos\frac{3x}{2} - \ldots - \cos\frac{(2n+1)x}{2}\\
&= \cos\frac{x}{2} - \cos\frac{(2n+1)x}{2}.
\end{aligned}$$

Applying the difference of two cosines formula (3.50) to the right-hand side, we get

$$2\sin\frac{x}{2}\cdot S_n = 2\sin\frac{\left(\frac{1}{2}+\frac{2n+1}{2}\right)x}{2}\cdot\sin\frac{\left(\frac{2n+1}{2}-\frac{1}{2}\right)x}{2} = 2\sin\frac{(n+1)x}{2}\cdot\sin\frac{nx}{2}$$

Solving this for S_n, we obtain the requested formula.

The proof is completed.

Problem 129 Evaluate $A = \frac{\cos\frac{\pi}{4}}{2} + \frac{\cos\frac{2\pi}{4}}{2^2} + \ldots + \frac{\cos\frac{\pi n}{4}}{2^n}$.

Solution Suppose that we have a complex number $A + iB$, such that

$A = \frac{\cos\frac{\pi}{4}}{2} + \frac{\cos\frac{2\pi}{4}}{2^2} + \ldots + \frac{\cos\frac{\pi n}{4}}{2^n}$

and

$$B = \frac{\sin\frac{\pi}{4}}{2} + \frac{\sin\frac{2\pi}{4}}{2^2} + \ldots + \frac{\sin\frac{\pi n}{4}}{2^n}.$$

Assuming that B is the imaginary part of the complex number, let us multiply it by i and add it to the expression for A:

$$A + iB = \tfrac{1}{2}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right) + \tfrac{1}{2^2}\left(\cos\tfrac{2\pi}{4} + i\sin\tfrac{2\pi}{4}\right) + \ldots + \tfrac{1}{2^n}\left(\cos\tfrac{n\pi}{4} + i\sin\tfrac{n\pi}{4}\right).$$

Applying Euler's formula to each quantity inside parentheses, we get

$$A + iB = \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) + \frac{1}{2^2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^2 + \ldots + \frac{1}{2^n}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n.$$

Next, we can see that the right side of the complex number is a geometric series with the first term and common ratio equal to

$$\frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

Therefore, using the formula for the sum of geometric series, we can rewrite our complex number as

$$A + iB = \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\frac{\left(1 - \frac{1}{2^n}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n\right)}{\left(1 - \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)}.$$

Applying De Moivre's formula to this again and using the fact that $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, we have the following complex number:

$$A + iB = \frac{1}{2\sqrt{2}}(1+i)\frac{\left(1 - \frac{1}{2^n}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n\right)}{\left(1 - \frac{1}{2\sqrt{2}} - \frac{i}{2\sqrt{2}}\right)}$$

Rationalizing the denominator and extracting the real part of $A + iB$ in the expression above, we obtain that

$$A = \frac{\left(\sqrt{2}-1\right)\left(2^n - \cos\frac{\pi n}{4}\right) + \sqrt{2}\sin\frac{\pi n}{4}}{2^n\left(5 - 2\sqrt{2}\right)}.$$

Answer $A = \frac{\left(\sqrt{2}-1\right)\left(2^n - \cos\frac{\pi n}{4}\right) + \sqrt{2}\sin\frac{\pi n}{4}}{2^n\left(5 - 2\sqrt{2}\right)}$.

3.6 Trigonometric Solution of Cubic Equations: Casus Irreducibilis

Consider a cubic equation $x^3 + px + q = 0$. Let us look for the solutions in the form

$$x = A\cos\varphi \tag{3.51}$$

Substituting (3.51) into the cubic equation we obtain

$$A^3\cos^3\varphi + Ap\cos\varphi = -q \tag{3.52}$$

Let us recall the formula for the cosine of a triple angle:

$$\cos 3\varphi = 4\cos^3\varphi - 3\cos\varphi \tag{3.53}$$

and compare (3.52) and (3.53). The idea is to find such values of the parameter p at which the left side of (3.52) can be written as $B\cos 3\varphi$, where B is some constant. Using the method of the undetermined coefficients, we obtain the following:

$$\begin{cases} A^3 = 4 \\ Ap = -3 \end{cases} \Rightarrow A = \sqrt{\frac{-4p}{3}} \quad \text{and} \quad A^2 = -\frac{4p}{3}, \quad p = -\frac{3A^2}{4}.$$

Because the value of A must be a real number, then it follows from $A = \sqrt{\frac{-4p}{3}}$ that this substitution can be used only for $p < 0$. We can substitute these in (3.52):

$$A\left(A^2\cos^3\varphi + p\cos\varphi\right) = -q$$
$$A\left(-\frac{4p}{3}\cos^3\varphi + p\cos\varphi\right) = -q$$
$$-\frac{Ap}{3}\left(4\cos^3\varphi - 3\cos\varphi\right) = -q$$

Substituting p in terms of A into the last equation, we have $\frac{A^3}{4}\cos 3\varphi = -q$ or

$$\cos 3\varphi = -\frac{4q}{A^3}$$
$$3\varphi = \pm\arccos\left(-\frac{4q}{A^3}\right) + 2\pi\cdot n$$
$$\varphi = \pm\frac{1}{3}\cdot\arccos\left(-\frac{4q}{A^3}\right) + \frac{2\pi}{3}\cdot n, \quad n \in Z$$

The last formula gives us three distinct values of the angle (solutions) within each revolution:

$$n = 0, \quad \varphi_1 = \frac{1}{3}\arccos\left(-\frac{4q}{A^3}\right)$$
$$n = 1, \quad \varphi_2 = \frac{1}{3}\arccos\left(-\frac{4q}{A^3}\right) + \frac{2\pi}{3}$$
$$n = 2, \quad \varphi_3 = \frac{1}{3}\arccos\left(-\frac{4q}{A^3}\right) + \frac{4\pi}{3}$$
$$n = 3, \quad \varphi_4 = \frac{1}{3}\arccos\left(-\frac{4q}{A^3}\right) + 2\pi$$

You can see that the first and the last angle are the same and they differ by 2π. Now we can obtain three different values of x:

$$x_1 = A\cos\varphi_1, x_2 = A\cos\varphi_2, x_3 = A\cos\varphi_3$$

Do you remember how we solved Problem 83 $(27x^3 + 54x^2 + 27x + 1 = 0)$ in Chapter 2? We reduced it to a new one, $y^3 - 3y - 1 = 0,\ p = -3,\ q = -1.$

Furthermore, this problem can be solved using a trigonometric approach because $p = -3 < 0$. Thus, we can evaluate that $A = 2$ and that

$$\varphi = \pm\frac{1}{3}\cdot\arccos\left(-\frac{4\cdot(-1)}{2^3}\right) + \frac{2\pi}{3}\cdot n,$$
$$\varphi = \pm\frac{1}{3}\cdot\arccos\left(\frac{1}{2}\right) + \frac{2\pi}{3}\cdot n$$
$$\varphi = \pm\frac{\pi}{9} + \frac{2\pi}{3}\cdot n$$

We can see that for $n = 0$, $n = 1$, and $n = 2$ we would have three distinct values of φ: $\varphi_1 = \frac{\pi}{9}, \varphi_2 = \frac{7\pi}{9}, \varphi_3 = \frac{13\pi}{9}$.

Remark 1 This method is efficient for finding solutions of $x^3 + px + q = 0$ if $p < 0, \frac{D}{4} = \frac{q^2}{4} + \frac{p^3}{27} < 0$.

Otherwise, try to use other methods.

Remark 2 Trigonometric substitution such as letting $x = \cos\varphi, y = \sin\varphi$ is very useful in solving some nonstandard problems. Some of the ideas can be found in the following sections.

3.7 Parameterized Form of a Curve

Sometimes a curve or a trajectory is given in a parameterized form, for example as $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ in R^2 or as $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ in R^3, where t is a parameter. The parameter t is often considered as time in physics problems, so a trajectory can be visualized as the set of ordered pairs $(x(t), y(t))$ at each time t. In order to find out more about a trajectory, we can graph the ordered pairs by hand, on a calculator or a computer. Sometimes we can eliminate t from the equations and often obtain a trajectory as an implicit curve, $F(x, y) = 0$ or $F(x, y, z) = 0$.

Let us consider several examples that students usually learn in calculus:

Example 1 We are given a trajectory $\begin{cases} x \pm c = a\cos t \\ y \pm d = b\sin t \end{cases}$.

If we square the left and right sides of each equation and then add the left and the right sides and apply a Pythagorean identity, we would obtain an implicit curve (ellipse) in the form of a conic:

$$\frac{(x \pm c)^2}{a^2} + \frac{(y \pm d)^2}{b^2} = 1.$$

An ellipse will become a circle if $a = b$.

The next example will present an explicit curve (line) also given by its parameterized form:

Example 2 $\begin{cases} x = t - 1 \\ y = 3t + 1 \end{cases}$.

The parameter t (time) can be eliminated from the system if we use the first equation substituted into the second one:

$$
\begin{aligned}
&t = x + 1\\
&y = 3(x + 1) + 1\\
&y = 3x + 4
\end{aligned}
$$

Clearly, now we obtained a line in its slope-intercept form.

Problem 130 Eliminate the parameter t from the system $\begin{cases} x = \tan t + 3 \\ y = \cot t - 1 \end{cases}.$

Solution We can rewrite the system as follows:
Multiplying the left sides and the right sides of the equations and using the property that $\tan t \cdot \cot t = 1$, we obtain

$$(x - 3)(y + 1) = 1$$

or

$$y = \frac{1}{x - 3} - 1.$$

Answer $y = \frac{1}{x-3} - 1.$

Often the parameter cannot be eliminated. There are many examples of such systems. For example, $\begin{cases} x = \cos t - t \\ y = \sin t + 3t \end{cases}.$

Sometimes we need to find a parametric curve from a given implicit or explicit curve. For example, working backwards, we can find that $(x - 1)^2 + (y + 2)^2 = 9$ (equation of a circle with center (1,−2) and radius 3) can be parameterized as

$$
\begin{aligned}
x - 1 &= 3\cos\varphi\\
y + 2 &= 3\sin\varphi
\end{aligned}
$$

This idea will be used in solving some nonstandard problems later in the text.

Some differential equations can be solved using the introduction of an additional variable.

Let us consider the following problem.

Problem 131 Solve the equation $\left(y'\right)^3 - 3y' + 2 = 0,\ \ y' = \frac{dy}{dx}.$

Solution At first glance, this is a nonlinear differential equation of the first order. However, there is no dependent variable there, only its derivative.

If we denote

$$y' = p,$$

we can rewrite the given equation as a polynomial equation in variable p:

$$p^3 - 3p + 2 = 0.$$

This polynomial equation has the following zeros:

$$\begin{array}{lcl} p = 1 & & p = -2 \\ y' = 1 & \text{and} & y' = -2 \\ y = x + C_1 & & y = -2x + C_2 \end{array}$$

Answer $y = x + C_1$; $y = -2x + C_2$.

What if I change the previous problem a little bit and give you the following problem?

Problem 132 Solve the equation $\left(y'\right)^3 - 3y' + 2 + x = 0$, $y' = \frac{dy}{dx}$, $y = y(x)$.

Solution If we denote $y' = p$, then we will have a polynomial equation in two variables, p and x:

$$p^3 - 3p + 2 + x = 0$$

from which we can express x in terms of p as

$$x = 3p - p^3 - 2$$

Wouldn't it be nice to find out how y depends on p? Then we would obtain the parameterized form of the solution!

If $y' = p \Rightarrow \frac{dy}{dx} = p$ which can also be written as

$$dy = pdx$$

How can we use this relationship?

Can we find dx? Yes. Let us differentiate both sides of the parameterized formula for x:

$$dx = \left(3 - 3p^2\right)dp$$

Substituting this into the formula for dy we obtain

$$dy = (3p - 3p^3)dp$$
$$y = \frac{3p^2}{2} - \frac{3p^4}{4} + C$$

Finally, we can put together both parameterized equations and obtain the solution to the given nonlinear differential equation:

Answer $\begin{cases} x = 3p - p^3 - 2 \\ y = \dfrac{3p^2}{2} - \dfrac{3p^4}{4} + C \end{cases}$

It would be interesting to solve a problem in which we do not have an independent variable.

Problem 133 Solve the equation $\left(y'\right)^3 - 3y' + 2 + y = 0,\ \ y' = \frac{dy}{dx}, y = y(x)$.

Solution In this problem we can again introduce a new variable:

$$y' = p$$

Then express y in terms of p:

$$y = 3p - p^3 - 2$$
$$dy = (-3p^2 + 3)dp$$
$$dx = \frac{dy}{p}$$
$$dx = \left(-3p + \frac{3}{p}\right)dp$$
$$x = -\frac{3p^2}{2} + 3\ln|p| + C$$

This parameterized solution is shown in Figure 3.31.

Answer $\begin{cases} x = -\dfrac{3p^2}{2} + 3\ln|p| + C \\ y = 3p - p^3 - 2, \qquad p \neq 0 \end{cases}.$

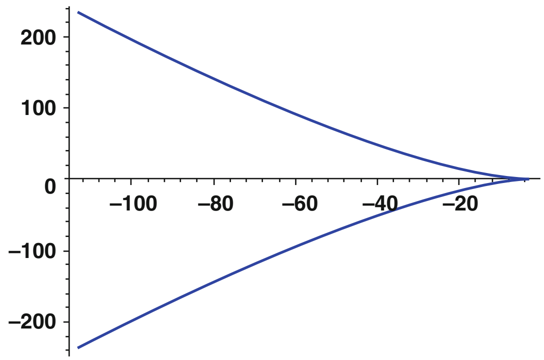


Figure 3.31 Sketch for Problem 133

3.8 Nonstandard Trigonometric Equations and Inequalities

In this section, I demonstrate some ideas for attacking and solving challenging trigonometric equations. I am sure that after reading this section you will find that many complex problems do not look complex anymore.

Problem 134 Solve the equation $\cos 6x + \sin \frac{5x}{2} = 2$.

Solution I noticed that usually students try to rewrite the left side as a product of two trig functions. However, that approach is good only if the right side is zero. For anything else it would not help much. Instead, focus on the boundary of the cosine and sine functions. Thus, the following is true:

$$\begin{cases} \cos 6x \le 1 \\ \sin \dfrac{5x}{2} \le 1 \end{cases} \Rightarrow \cos 6x + \sin \frac{5x}{2} \le 2$$

Therefore, the left side of the equation is less than or equal to 2 and the right side is 2. Hence, this equation can have a solution iff both $\cos 6x = 1$ and $\sin \frac{5x}{2} = 1$ simultaneously.

Then we need to solve the following system:

$$\begin{cases} \cos 6x = 1 \\ \sin \dfrac{5x}{2} = 1 \end{cases} \Leftrightarrow \begin{cases} 6x = 2\pi n \\ \dfrac{5x}{2} = \dfrac{\pi}{2} + 2\pi m \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{\pi}{3} \cdot n, \quad n \in Z \\ x = \dfrac{\pi}{5} \cdot (4m+1), \; m \in Z \end{cases} \tag{3.54}$$

It would be wrong to state that we solved the equation by giving the answer above. You can see that the first value of x depends on an integer n and the second x depends on an integer m. These integers are changing independently; moreover, the solutions are given by two formulas, but we need only one solution.

Let us equate the right sides of the equations in the last system:

$$\frac{\pi}{3} \cdot n = \frac{\pi}{5} \cdot (4m + 1)$$
$$5n = 3(4m + 1)$$

The following equation must be solved in integers:

$$5n = 12m + 3$$

The right side is divisible by 3; then we need to replace

$$n = 3k$$

and substitute it back into the equation

$$15k = 3(4m + 1)$$

or

$$5k = 4m + 1 \tag{3.55}$$

Next, we can think of it this way: the right side is odd, but 5 k can be either odd or even, so in order to have solutions we need to select only odd values of k; then

$$k = 2u + 1 \tag{3.56}$$

Substituting (3.56) into (3.55) we obtain the following chain of true relationships:

$$\begin{aligned} &10u + 5 = 4m + 1 \\ &5u = 2m - 2 \\ &u = 2t \\ &10t = 2m - 2 \\ &5t = m - 1 \end{aligned}$$

Finally, we obtain that

$$m = 5t + 1, \quad t \in Z$$

This expression for integer m can be substituted into the second solution of (3.54) to give us the final answer:

$$x = \frac{\pi}{5} \cdot (4m + 1) = \frac{\pi}{5} \cdot (4(5t + 1) + 1) = \frac{\pi}{5}(20t + 5) = \pi(4t + 1) = \pi + 4\pi \cdot t, t \in Z.$$

Answer $x = \pi + 4\pi t, \quad t \in Z.$

Problem 135 Solve the equation $\cos 6x + \sin 6x = 2$.

Solution If you try to do this problem similarly to the previous one, you would not succeed because cosine and sine depend on each other. You need to use an auxiliary argument and rewrite the left side as we did in Section 3.3.4:

$$\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos 6x + \frac{1}{\sqrt{2}}\sin 6x\right) = \sqrt{2}\sin\left(\frac{\pi}{4} + 6x\right)$$

Using boundedness of the sine function, we conclude that the equation cannot have any solutions because the right side (number 2) is greater than the left side ever gets.

In the following problem, you can use the boundary of sine and cosine again.

Problem 136 Find real x and y that satisfy the equation $\cos x + \cos y = \frac{3}{2} + \cos(x+y)$.

Solution We will rewrite the left side as a product and use the double-angle formula for the cosine function on the right:

$$2\cos\frac{x+y}{2}\cdot\cos\frac{x-y}{2} = \frac{1}{2} + (1 + \cos(x+y))$$
$$2\cos\frac{x+y}{2}\cdot\cos\frac{x-y}{2} = \frac{1}{2} + 2\cos^2\frac{x+y}{2}$$

Multiplying both sides by 2 we obtain the equation below, which is equivalent to the original equation:

$$4\cos\frac{x+y}{2}\cdot\cos\frac{x-y}{2} = 1 + 4\cos^2\frac{x+y}{2} \tag{3.57}$$

On the other hand, using the trinomial square formula, we can write

$$\left(2\cos\frac{x+y}{2} - \cos\frac{x-y}{2}\right)^2 = 4\cos^2\frac{x+y}{2} + \cos^2\frac{x-y}{2} - 4\cos\frac{x+y}{2}\cdot\cos\frac{x-y}{2}$$

which is the same as

$$\begin{aligned}4\cos\frac{x+y}{2}\cdot\cos\frac{x-y}{2} = {} & 4\cos^2\frac{x+y}{2} + \cos^2\frac{x-y}{2} \\ & - \left(2\cos\tfrac{x+y}{2} - \cos\tfrac{x-y}{2}\right)^2\end{aligned} \tag{3.58}$$

Since the left sides of (3.57) and (3.58) are the same the right sides must be the same:

$$4\cos^2\frac{x+y}{2}+\cos^2\frac{x-y}{2}-\left(2\cos\frac{x+y}{2}-\cos\frac{x-y}{2}\right)^2=1+4\cos^2\frac{x+y}{2}$$

After further simplifications, we obtain

$$\cos^2\frac{x-y}{2}-1=\left(2\cos\frac{x+y}{2}-\cos\frac{x-y}{2}\right)^2 \tag{3.59}$$

If we look at (3.59) as $f(x)=g(x)$, then we know that $f(x)\le 0$ and $g(x)\ge 0$ because of the boundedness of the cosine functions.

In order to have a possible solution both sides must take the same value of zero. The following system has to be solved:

$$\begin{cases} 2\cos\dfrac{x+y}{2}-\cos\dfrac{x-y}{2}=0 \\ \cos^2\dfrac{x+y}{2}=1 \end{cases} \tag{3.60}$$

System (3.60) is now equivalent to two cases:

Case 1

$$\begin{cases} \cos\dfrac{x-y}{2}=1 \\ \cos\dfrac{x+y}{2}=\dfrac{1}{2} \end{cases} \Rightarrow \begin{cases} \dfrac{x-y}{2}=2\pi n \\ \dfrac{x+y}{2}=\pm\dfrac{\pi}{3}+2\pi m \end{cases}$$

$$\begin{cases} x-y=4\pi n \\ x+y=\pm\dfrac{2\pi}{3}+4\pi m \end{cases} \begin{cases} x_1=\pm\dfrac{\pi}{3}+2\pi(m+n) \\ y_1=\pm\dfrac{\pi}{3}+2\pi(m-n) \end{cases}$$

Case 2

$$\begin{cases} \cos\dfrac{x-y}{2}=-1 \\ \cos\dfrac{x+y}{2}=-\dfrac{1}{2} \end{cases} \Rightarrow \begin{cases} \dfrac{x-y}{2}=\pi(2n+1) \\ \dfrac{x+y}{2}=-\dfrac{2\pi}{3}+2\pi m \end{cases} \begin{cases} x_2=\pm\dfrac{2\pi}{3}+\pi(2m+2n+1) \\ y_2=\pm\dfrac{2\pi}{3}+\pi(2m-2n-1) \end{cases}$$

Problem 137 Solve the equation $2+2(\sin y+\cos y)\sin x=\cos 2x$.

Solution Let us replace the cosine using a double-angle formula:

$$\begin{aligned}
\cos 2x &= 1 - 2\sin^2 x \\
2 + 2(\sin y + \cos y)\sin x &= 1 - 2\sin^2 x \\
2\sin^2 x + 2(\sin y + \cos y)\sin x + 1 &= 0
\end{aligned} \tag{3.61}$$

Equation (3.61) can be considered as quadratic in $\sin x$. Using $\frac{D}{4}$ formula we obtain that

$$\frac{D}{4} = (\sin y + \cos y)^2 - 2 = \sin 2y - 1$$

Therefore, (3.61) will have solutions iff $\sin 2y - 1 \geq 0 \Rightarrow \sin 2y \geq 1$. Since $\sin 2y$ cannot be greater than 1, the only option for this equation to have a real solution is

$$\begin{aligned}
\sin 2y &= 1 \\
y &= \frac{\pi}{4} + \pi \cdot n
\end{aligned} \tag{3.62}$$

Hence, the discriminant of the quadratic formula is zero and (3.61) has only one zero:

$$\sin x = \frac{-(\sin y + \cos y)}{2} \tag{3.63}$$

If we substitute (3.62) into (3.63) we will obtain two values of $\sin x$:

$$\sin x = \frac{\sqrt{2}}{2} \text{ or } \sin x = -\frac{\sqrt{2}}{2}$$

Uniting solutions of these two equations we obtain that

$$x = \pm\frac{\pi}{4} + \pi m.$$

Answer $x = \pm\frac{\pi}{4} + \pi m, y = \frac{\pi}{4} + \pi n, \quad n, m \in Z.$

Remark You can get the same answer for x if we attack (3.63) differently.
For example, we can combine $\sin y + \cos y$ as $\sqrt{2}\sin\left(\frac{\pi}{4} + y\right)$ and then rewrite (3.63) with the use of (3.62) as

$$\begin{aligned}
\sin x &= \frac{\sqrt{2}}{2}\sin\left(\frac{\pi}{4} + y\right) \\
\sin x &= \frac{\sqrt{2}}{2}\sin\left(\frac{\pi}{4} + \frac{\pi}{4} + \pi n\right) \\
\sin x &= \frac{\sqrt{2}}{2}\sin\left(\frac{\pi}{2} + \pi n\right) \\
\sin x &= \pm\frac{\sqrt{2}}{2}
\end{aligned}$$

Problem 138 (Suprun) Solve the equation $\sin^6 x + \cos^6 x = a\sin 4x$ for each possible value of a parameter a.

Solution Consider the left side as the sum of two cubes:

$$\sin^6 x + \cos^6 x = (\sin^2 x)^3 + (\cos^2 x)^3 = (\sin^2 x)^2 - \sin^2 x\cos^2 x + (\cos^2 x)^2$$
$$= 1 - 3\sin^2 x\cos^2 x = 1 - \frac{3}{4}\cdot \sin^2 2x$$

Next, we will apply the double-angle formula as $\sin^2 2x = \frac{1-\cos 4x}{2}$:

This will make the left side of our problem

$$1 - \frac{3}{4}\sin^2 2x = 1 - \frac{3}{8}(1 - \cos 4x) = \frac{5}{8} + \frac{3}{8}\cos 4x.$$

Putting together the left and the right sides we obtain the following equation to solve; we will use the technique of adding an auxiliary argument:

$$\frac{5}{8} + \frac{3}{8}\cos 4x = a\sin 4x$$

or

$$a\sin 4x - \frac{3}{8}\cos 4x = \frac{5}{8}$$
$$\sqrt{a^2 + \frac{9}{64}}\cdot \sin(4x + \varphi) = \frac{5}{8}, \quad \varphi = -\arccos\frac{a}{\sqrt{a^2 + \frac{9}{64}}} = -\arccos\frac{8a}{\sqrt{64a^2 + 9}}$$

$$\sin(4x + \varphi) = \frac{5}{\sqrt{64a^2 + 9}} \tag{3.64}$$

This equation will have a real solution if and only if the right side is less than or equal to one:

$$\sqrt{64a^2 + 9} \geq 5$$
$$64a^2 + 9 \geq 25$$
$$64a^2 \geq 16$$
$$a^2 \geq \frac{1}{4}$$
$$|a| \geq \frac{1}{2}$$

Therefore, if $-\frac{1}{2} < a < \frac{1}{2}$, (3.64) has no real solutions.

If $a \le -\frac{1}{2}$ or $a \ge \frac{1}{2}$, then the solution of (3.64) can be found as

$$x = \frac{(-1)^n}{4}\arcsin\frac{5}{\sqrt{64a^2+9}} + \frac{1}{4}\arccos\frac{8a}{\sqrt{64a^2+9}} + \frac{\pi n}{4}.$$

Sometimes a problem that at first glance does not even have any trigonometric functions can be solved with the use of trigonometric equations. Also, there are many trigonometry problems that contain a mixture of different functions with restricted domains, such as logarithmic functions. For your consideration, here are some interesting problems.

Problem 139 Find maximum and minimum of $A = xw + yz - x + w + 2z - 61$ if x, y, z, and w satisfy the system

$$\begin{cases} x^2 + y^2 + 2x + 4y - 20 = 0 \\ z^2 + w^2 - 2w - 143 = 0 \end{cases}$$

Solution Completing the square in both equations we obtain

$$\begin{aligned} (x+1)^2 + (y+2)^2 &= 25 \\ z^2 + (w-1)^2 &= 144 \end{aligned} \tag{3.65}$$

The function of four variables will also be rewritten as

$$A = (x+1)(w-1) + z(y+2) - 60 \tag{3.66}$$

Let us try to parameterize (3.65) and (3.66).

Formula (3.65) represents two different circles. Therefore we can introduce

$$\begin{cases} x + 1 = 5\cos\varphi \\ y + 2 = 5\sin\varphi \\ z = 12\cos\psi \\ w - 1 = 12\sin\psi \end{cases} \tag{3.67}$$

Substituting (3.67) into (3.66) we get

$$\begin{aligned} A &= 5\cos\varphi \cdot 12\sin\psi + 12\cos\psi \cdot 5\sin\varphi - 60 \\ A &= 60\sin(\varphi + \psi) - 60 \end{aligned}$$

Because sine is a bounded function, then $A_{\min} = -120,\ \ A_{\max} = 0$.

Answer $A_{\min} = -120,\ \ A_{\max} = 0$.

Problem 140 (Chirsky) Solve the equation $\sin x^{-\sin x} - 1 = \cot^2 x$.

Solution First, the left side is defined only for positive values of the sine function, $\sin x > 0$.

Next, we will move negative one to the right side and apply a trigonometric identity, so that we have

$$(\sin x)^{-\sin x} = (\sin x)^{-2}$$
$$\begin{cases} (\sin x)^{2-\sin x} = 1 \\ \sin x > 0\ (\sin x \le 1) \end{cases}$$
$$\sin x = 1 \Rightarrow x = \frac{\pi}{2} + 2\pi n.$$

Answer $x = \frac{\pi}{2} + 2\pi n$.

Problem 141 (Lidsky) Let $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \frac{\pi}{2}$. Prove that $\tan \alpha_1 < \dfrac{\sin \alpha_1 + \sin \alpha_2 + \ldots + \sin \alpha_n}{\cos \alpha_1 + \ldots + \cos \alpha_n} < \tan \alpha_n$.

Proof What do we know about the tangent function? We know that it is a periodic function that is monotonically increasing in the first quadrant. Because all angles are restricted by the first quadrant, we know that the following is true for tangent values:

$$\tan \alpha_1 \le \ldots \le \tan \alpha_i \le \ldots \tan \alpha_n$$

We also know that for any angle $0 < \alpha_i < \frac{\pi}{2}$ of the first quadrant, the cosine function is positive and we can state that the following is valid:

$$\cos \alpha_i > 0$$
$$\tan \alpha_1 \le \tan \alpha_i \le \tan \alpha_n, \quad 0 < i < n \tag{3.68}$$

Let us multiply (3.68) by $\cos \alpha_i > 0$:

$$\tan \alpha_1 \cdot \cos \alpha_i \le \tan \alpha_i \cdot \cos \alpha_i \le \tan \alpha_n \cdot \cos \alpha_i \tag{3.69}$$

Inequality (3.69) is true for any angle alpha from the first quadrant. Thus we can write such an inequality for each possible α_i, $i = 1, 2, \ldots, n$:

$$\begin{gathered} \tan \alpha_1 \cdot \cos \alpha_1 \le \tan \alpha_1 \cdot \cos \alpha_1 \le \tan \alpha_n \cdot \cos \alpha_1 \\ \tan \alpha_1 \cdot \cos \alpha_2 \le \tan \alpha_2 \cdot \cos \alpha_2 \le \tan \alpha_n \cdot \cos \alpha_2 \\ \ldots \\ \tan \alpha_1 \cdot \cos \alpha_n \le \tan \alpha_n \cdot \cos \alpha_n \le \tan \alpha_n \cdot \cos \alpha_n \end{gathered}$$

If we add all these double inequalities, replace $\tan \alpha_i \cdot \cos \alpha_i = \sin \alpha_i$ in the middle and factor common factors on the left and right, we obtain the following true inequality:

$$\begin{aligned}\tan\alpha_1\cdot\left(\cos\alpha_1+\ldots+\cos\alpha_n\right) &\le \sin\alpha_1+\ldots+\sin\alpha_n\\ &\le \tan\alpha_n\cdot\left(\cos\alpha_1+\ldots+\cos\alpha_n\right)\end{aligned}$$

Dividing all sides of this double inequality by the positive quantity inside parentheses will complete our proof. Therefore,

$$\tan\alpha_1<\frac{\sin\alpha_1+\sin\alpha_2+\ldots+\sin\alpha_n}{\cos\alpha_1+\ldots+\cos\alpha_n}<\tan\alpha_n.$$

Problem 142 Solve the equation $3(\log_2\sin x)^2+\log_2(1-\cos 2x)=2$.

Solution Because $1-\cos 2x=2\sin^2 x\ge 0$, and considering restrictions that come from the first term, we need to impose a condition that $\sin x>0$.

Let us simplify the second logarithm:

$$\log_2(1-\cos 2x)=\log_2\left(2\sin^2 x\right)=1+2\log_2\sin x.$$

Now the equation can be rewritten as

$$\begin{aligned}&3(\log_2(\sin x))^2+2\log_2(\sin x)-1=0\\ &y=\log_2(\sin x)\\ &3y^2+2y-1=0\end{aligned}$$

$$\begin{array}{lll} 1.\ y=-1 & \text{or} & 2.\ y=\dfrac{1}{3}\\[2mm] \log_2(\sin x)=-1 & & \log_2(\sin x)=\dfrac{1}{3}\\[2mm] \sin x=\dfrac{1}{2} & & \sin x=2^{\frac{1}{3}}\\[2mm] x=(-1)^n\dfrac{\pi}{6}+\pi n, n\in Z & & \varnothing \end{array}$$

3.9 Homework on this chapter

1. Solve the inequality $y-\frac{1}{|\cos x|}-\sqrt{1-y-x^2}\ge 0$.

   Solution: Let us rewrite this inequality as

$$\sqrt{1-y-x^2}\le y-\frac{1}{|\cos x|}$$

   Because $1-y-x^2\ge 0\Rightarrow 1-x^2\ge y$. This can be rewritten as

$$y \le 1 - x^2 \tag{3.70}$$

On the other hand, because the square root is positive, then (3.70) is true, if $1 - \frac{1}{|\cos x|} \ge 1$. This can be written as

$$y \ge \left| \frac{1}{\cos x} \right| \ge 1 \tag{3.71}$$

Combining (3.70) and (3.71) we obtain

$$1 \le y \le 1 - x^2$$

The last double inequality is true iff $x = 0, y = 1$.
Answer: $x = 0,\ y = 1$.

2. Solve the equation $(\tan x)^{\sin x} = (\cot x)^{\cos x}$.

   Solution: Making the replacement $\cot x = \frac{1}{\tan x}$ the given equation can be rewritten as $(\tan x)^{\sin x} = (\tan x)^{-\cos x}$

   If $\tan x < 0$ but $\sin x, \cos x$ are fractions less than one, then this equation does not make sense.

   If $\tan x = 0$ then $\sin x = 0$ and again the equation does not make sense.

   If $\tan x > 0$ but $\tan x \ne 1$ then $\sin x = -\cos x$ from which $\tan x < 0$ which is a contradiction.

   Finally, if $\tan x = 1$ then $x = \frac{\pi}{4} + \pi n, \quad n \in Z$.

3. Solve the equation $\log \sin x + \log \sin 5x + \log \cos 4x = 0$.

   Solution: Let us rewrite the equation as

$$\sin x \cdot \sin 5x \cdot \cos 4x = 1$$

Recalling the material of Chapter 1, all functions in the product are bounded and their absolute value must be less than or equal to one. Moreover, the original equation involves logarithms and each quantity that we are taking a logarithm of must be positive. Therefore, in order for this equation to have solutions, the following must be true:

$$\begin{cases} \sin x \sin 5x \cos 4x = 1 \\ |\sin x| = |\sin 5x| = |\cos 4x| = 1 \\ \sin x > 0 \\ \sin 5x > 0 \\ \cos 4x > 0 \end{cases}$$

The system can be rewritten as

$$\begin{cases} \sin x = 1 \\ \sin 5x = 1 \\ \cos 4x = 1 \end{cases}$$

Solving the first equation we obtain $x = \frac{\pi}{2} + 2\pi n$. We will substitute it into the second and third equations:

$$\sin\left(5\left(\frac{\pi}{2} + 2\pi n\right)\right) = \sin\frac{\pi}{2} = 1, \quad \cos\left(4\left(\frac{\pi}{2} + 2\pi n\right)\right) = \cos 0^\circ = 1$$

which gives us the answer $x = \frac{\pi}{2} + 2\pi n$.

4. Solve the equation $\sin^2 3x + \cos^2 2x = 1$.

   Hint: Reduce the power and then use the difference of two cosines formula to obtain $\sin 5x \cdot \sin x = 0$.

   Answer: $x = \frac{\pi n}{5}, n \in Z$.

5. Solve the equation $\sin x + \sin 2x + 2\sin x \sin 2x = 2\cos x + \cos 2x$.

   Solution: Let us rewrite the product above as the difference of two cosines:

$$\begin{aligned} &\sin x + \sin 2x + \cos x - \cos 3x = 2\cos x + \cos 2x \\ &\sin x + \sin 2x - \cos x - \cos 2x - \cos 3x = 0 \\ &\sin x + \sin 2x - (\cos x + \cos 3x) - \cos 2x = 0 \end{aligned}$$

Next, rewrite the sum inside parentheses as a product:

$$\sin x + 2\sin x \cos x - 2\cos 2x \cos x - \cos 2x = 0$$

Now combine the first two and the last two terms and factor out common factors by grouping:

$$\begin{aligned} &\sin x(1 + 2\cos x) - \cos 2x(1 + 2\cos x) = 0 \\ &(\sin x - \cos 2x)(1 + 2\cos x) = 0 \end{aligned}$$

There are two cases.

Case 1 $\sin x - \cos 2x = 0$

Using a double-angle formula we get the quadratic equation below:

$$\sin x - 1 + 2\sin^2 x = 0$$

$$2y^2 + y - 1 = 0$$

$$\sin x = -1 \quad \text{or} \quad \sin x = \frac{1}{2}$$

Case 2 $1 + 2\cos x = 0$.

$$\cos x = -\frac{1}{2}$$
$$x = \pm\frac{2\pi}{3} + 2\pi k, k \in Z.$$

Answer: $x = \pm\frac{2\pi}{3} + 2\pi n,\ \ x = -\frac{\pi}{2} + 2\pi m,\ \ x = (-1)^k\frac{\pi}{6} + \pi k,\ \ m, n, k \in Z.$

6. Solve the equation $\sin 4x = \tan x$.

   Hint: Use the double-angle formula twice and rewrite tangent as the ratio of sine and cosine.

   Solution:

$$2\sin 2x\cos 2x = 4\sin x\cos x\cos 2x = \frac{\sin x}{\cos x}$$
$$\sin x \cdot (4\cos^2 x \cdot \cos 2x - 1) = 0$$
$$1.\ \sin x = 0 \quad \text{or} \quad 2.\ 4\cos^2 x \cdot \cos 2x - 1 = 0$$
$$x = \pi m \qquad 2 \cdot 2\cos^2 x \cdot \cos 2x - 1 = 0$$
$$2(1 + \cos 2x)\cos 2x - 1 = 0$$
$$2\cos^2 2x + 2\cos 2x - 1 = 0$$
$$\cos 2x = \frac{\sqrt{3}-1}{2}$$
$$x = \pm\frac{1}{2}\arccos\frac{\sqrt{3}-1}{2} + \pi n$$

7. Solve the equation $4\sin^4 x = 1 + 5\cos^2 x$.

   Hint: Reduce the power on the left and on the right and rewrite the equation in terms of cos2x.

   Answer: $x = \pm\frac{\pi}{3} + \pi n, n \in Z.$

8. Solve the equation $\sin x\cos x\cos 2x\cos 8x = -\frac{1}{4}\sin 12x$.

   Hint: Apply the formula for the sine of a double angle to both sides of the equation and then the formula for the product of sine and cosine on the left, and then factor.

   Answer: $x = \frac{\pi}{8} \cdot n, n \in Z.$

9. Solve $\cos 2x - 5\sin x - 3 = 0$.

   Hint: Rewrite cos2x in terms of sin$x$.

   Answer: $x = (-1)^n\frac{\pi}{6} + \pi k,\ \ k \in Z.$

10. Solve the equation $\frac{1}{\sqrt{3}-\tan x} - \frac{1}{\sqrt{3}+\tan x} = \sin 2x$.

    Hint: Substitute $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$.

Answer: $x = \frac{\pi}{4} + \frac{\pi}{2} \cdot n;\ \ x = \pi k,\ \ n, k \in Z.$

11. Solve $\sin x \sin 3x = -\sin 4x \sin 8x$.

    Answer: $x = \frac{\pi n}{5}; x = \frac{\pi k}{7},\ \ n, k \in Z.$

12. Solve the equation $\cot x - \tan x = \frac{\cos x - \sin x}{\frac{1}{2}\sin 2x}$.

    Hint: Replace tangent and cotangent of a single argument in terms of sine and cosine of a double argument.

    Solution: Because $\cot x = \frac{1+\cos 2x}{\sin 2x}$, $\tan x = \frac{1-\cos 2x}{\sin 2x}$, the equation becomes

$$\frac{2\cos 2x}{\sin 2x} = \frac{\cos x - \sin x}{\frac{1}{2}\sin 2x}, \qquad \sin 2x \neq 0$$
$$\cos^2 x - \sin^2 x = \cos x - \sin x$$
$$(\cos x - \sin x)(\cos x + \sin x - 1) = 0$$

Usually we have to equate each factor to zero, but the second factor cannot be zero because the denominator of the original equation is not zero.
   Therefore, the solution can be found by solving

$$\cos x - \sin x = 0$$
$$x = \frac{\pi}{4} + \pi n, \quad n \in Z.$$

13. Solve the equation $(\sin x + \cos x)^4 + (\sin x - \cos x)^4 = 3 - \sin 4x$.

    Answer: $x = \frac{\pi}{16}(1 + 4n),\ \ n \in Z.$

14. Solve the equation $(\cot x - 1)(1 + \sin 2x) = 1 + \cot x$.

    Hint: Use $\cot x = \frac{1}{\tan x}$ and $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$; denote $y = \tan x \neq 0$.

    Answer: $x = -\frac{\pi}{4} + \frac{\pi n}{2};\ \ x = \arctan 3 + \pi k,\ \ n, k \in Z.$

15. Solve $\sin^2 x - 2\sin x \cos x = 3\cos^2 x$.

    Hint: This is a homogeneous equation of second order.

    Answer: $x = \frac{\pi}{4} + \frac{\pi n}{2};\ \ x = \arctan 3 + \pi k,\ \ n, k \in Z.$

16. Solve $\cos 3x - \cos^2 x + \frac{3}{4}\sin 2x = 0$.

    Hint: Multiply the equation by 4 and then replace $4\cos^3 x = \cos 3x + 3\cos x$.

    Solution:

$$4\cos 3x - \cos 3x - 3\cos x + 3\sin 2x = 0$$
$$(\cos 3x - \cos x) + \sin 2x = 0$$

Rewrite the difference of cosines as the product and then factor out the common factor:

$$\begin{array}{ll} -2\sin 2x\sin x+\sin 2x=0 & \\ \sin 2x\cdot(1-2\sin x)=0 & \\ 1.\ \sin 2x=0 \quad \text{or} & 2.\ 1-2\sin x=0 \\ 2x=\pi n & \sin x=\dfrac{1}{2} \\ x=\dfrac{\pi}{2}n & x=(-1)^m\dfrac{\pi}{6}+\pi m \end{array}$$

Answer: $\frac{\pi}{2}\cdot n;\ \ (-1)^m\frac{\pi}{6}+\pi m,\ n,m\in Z.$

17. Find all solutions to $5\sin^2 x+2\sqrt{3}\sin x\cos x-\cos^2 x=2$ that satisfy the inequality $-\pi<x<\pi$.

Hint: Replace 2 on the right-hand side by a double trigonometric identity, then simplify and recognize a homogeneous equation of second order, and divide both sides by $\cos^2 x$. Denote $y=\tan x$. You will obtain the equation $3y^2+2\sqrt{3}y-3=0$ and $x=\frac{\pi}{6}+\pi n$. Next we will select only such solutions that belong to the given interval. There are at $n=-1$ and $n=0$.

Answer: $-\frac{5\pi}{6},\frac{\pi}{6}.$

18. Solve $\log_{\sqrt{2}\sin x}(1+2\cos^2 x)=2$.

Hint: Because this equation is logarithmic, make sure that you find restrictions on the independent variable.

Solution: First, we will find the restrictions $\begin{cases}\sqrt{2}\sin x>0\\ \sqrt{2}\sin x\neq 1\end{cases}.$

Second, applying the definition of the logarithm, the given equation can be rewritten as

$$\begin{aligned} &1+2\cos^2 x=\left(\sqrt{2}\sin x\right)^2\\ &1+2(1-\sin^2 x)=2\sin^2 x\\ &4\sin^2 x=3\\ &\sin x=\frac{\sqrt{3}}{2}\\ &x=(-1)^n\frac{\pi}{3}+\pi n. \end{aligned}$$

We selected only the positive root because of the restriction in the domain.

Answer: $x=(-1)^n\frac{\pi}{3}+\pi n, n\in Z.$

19. Solve the equation $3^{\sin 2x+2\cos^2 x}+3\cdot 9^{-\sin x(\cos x-\sin x)}=28$.

Hint: Note that $\sin 2x+2\cos^2 x=2(\sin x\cos x+\cos^2 x)$ and that $-\sin x(\cos x-\sin x)=1-(\sin x\cos x+\cos^2 x)$. Let $y=9^{\sin x\cos x+\cos^2 x}$.

Solution: In terms of the new variable our equation will be rewritten as

$$y + 3 \cdot 9 \cdot \frac{1}{y} = 28$$
$$y^2 - 28y + 27 = 0$$
$$y = 1 \quad \text{or} \quad y = 27.$$

For each y we will find the corresponding x value:

Case 1
$$9^{\sin x \cos x + \cos^2 x} = 1$$
$$\sin x \cos x + \cos^2 x = 0$$
$$\cos x(\sin x + \cos x) = 0$$
$$x = \frac{\pi}{2} + \pi n; \quad x = -\frac{\pi}{4} + \pi k.$$

Case 2
$$9^{\sin x \cos x + \cos^2 x} = 27$$
$$\sin x \cos x + \cos^2 x = \frac{3}{2}$$
$$2\cos^2 x - 2\sin x \cos x + 3\sin^2 x = 0$$
$$\varnothing \quad \text{no real solutions}$$

Answer: $\frac{\pi(2n+1)}{2}; \frac{(4k-1)\pi}{4}$.

20. Solve the equation $2\sqrt{1 - x^2}\sin^2 x = \sqrt{1 - x^2} - \cos 2x$.

Hint: Consider the restricted domain of the square root function.

Solution: Domain of the square root function is

$$1 - x^2 \geq 0 \Leftrightarrow -1 \leq x \leq 1. \tag{3.72}$$

Next, we will use a power reduction formula:

$$2\sin^2 x = 1 - \cos 2x$$

And factor the equation as

$$\cos 2x\left(\sqrt{1 - x^2} - 1\right) = 0$$
$$1. \quad \cos 2x = 0 \quad \text{or} \quad 2. \quad \sqrt{1 - x^2} = 1 \Rightarrow x = 0 \cdot$$

The first equation give us a solution

$$x = \frac{\pi}{4} + \frac{\pi}{2} \cdot n, \; n \in Z.$$

Since our solution must satisfy (3.72), the following must be valid:

$$-1 \leq \frac{\pi}{4} + \frac{\pi}{2} \cdot n \leq 1$$
$$n = -1, x = -\frac{\pi}{4}$$
$$n = 0, x = \frac{\pi}{4}$$

21. Solve the equation $(1+\tan^2 x)\sin x - \tan^2 x + 1 = 0$ given the condition $\tan x < 0$.

Solution: First, we will restrict the domain of the tangent function, so $x \neq \frac{\pi}{2} + \pi n$. Second, we will make substitution $1 + \tan^2 x = \frac{1}{\cos^2 x}$ and simplify:

$$\frac{1+\sin x - 2\sin^2 x}{\cos^2 x} = 0.$$

This equation is equivalent to the following system:

$$\begin{cases} (\sin x - 1)\left(\sin x + \dfrac{1}{2}\right) = 0 \\ \cos x \neq 0 \\ \tan x < 0 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} x = \dfrac{\pi}{6} + 2\pi n \\ x = \dfrac{7\pi}{6} + 2\pi k \end{array}\right. \\ \tan x < 0 \end{cases} \Leftrightarrow x = -\frac{\pi}{6} + 2\pi n.$$

Answer: $-\frac{\pi}{6} + 2\pi n, n \in Z.$

22. Solve the equation $\log_2(3 - \sin x) = \sin x$.

Solution: Because $\sin x \leq 1 \Rightarrow 3 - \sin x \geq 2$, the following is true:

$$\begin{aligned} &1 \geq \sin x = \log_2(3 - \sin x) \geq \log_2 2 = 1 \\ &\sin x = 1 \\ &x = \frac{\pi}{2} + 2\pi n \end{aligned}$$

Answer: $x = \frac{(4n+1)}{2} \cdot \pi, \quad n \in Z.$

23. Solve the inequality $1 - \sin x + \cos x < 0$.

Answer: $\frac{\pi}{2} + 2\pi n < x < \pi + 2\pi n.$

Solution: We can rewrite this inequality as

$$\begin{aligned} &\cos x - \sin x < -1 \\ &\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x < -\frac{1}{\sqrt{2}} \\ &\sin\left(\frac{\pi}{4} - x\right) < -\frac{1}{\sqrt{2}} \\ &\sin\left(x - \frac{\pi}{4}\right) > \frac{1}{\sqrt{2}} \\ &\frac{\pi}{4} + 2\pi n < x - \frac{\pi}{4} < \frac{3\pi}{4} + 2\pi n \\ &\frac{\pi}{2} + 2\pi n < x < \pi + 2\pi n \end{aligned}$$

24. Evaluate the partial sum $S_n = \cos\frac{\pi}{n} + \cos\frac{2\pi}{n} + \ldots + \cos\frac{(n-1)\pi}{n}$.

Solution: Let us multiply the partial sum by $2\sin\frac{\pi}{2n}$. Using identity (3.49) and the fact that sine is an odd function we will obtain the following:

$$\begin{aligned}2S_n\sin\frac{\pi}{2n} &= 2\sin\frac{\pi}{2n}\cos\frac{\pi}{n} + 2\sin\frac{\pi}{2n}\cos\frac{2\pi}{n} + \ldots + 2\sin\frac{\pi}{2n}\cos\frac{(n-1)\pi}{n}\\ &= \sin\frac{3\pi}{2n} - \sin\frac{\pi}{2n} + \sin\frac{5\pi}{2n} - \sin\frac{3\pi}{2n} + \ldots\\ &\quad + \sin\frac{(2n-3)\pi}{2n} - \sin\frac{(2n-5)\pi}{2n} + \sin\frac{(2n-1)\pi}{2n} - \sin\frac{(2n-3)\pi}{2n}.\end{aligned}$$

After simplification and canceling opposite terms, we have

$$2S_n\sin\frac{\pi}{2n} = -\sin\frac{\pi}{2n} + \sin\frac{(2n-1)\pi}{2n} = -\sin\frac{\pi}{2n} + \sin\left(\pi - \frac{\pi}{2n}\right) = 0$$

Considering the equation above, we notice that the required sum multiplied by the nonzero factor $\left(2\sin\frac{\pi}{2n} \neq 0\right)$ equals zero. Therefore the sum is zero.

Answer: 0.

25. Evaluate $S = \sin 10^\circ + \sin 20^\circ + \sin 30^\circ + \sin 40^\circ + \ldots + \sin 2020^\circ$

Hint: See Problem 129.

26. Prove that $f(x) = \tan x$ is an odd function.
27. Express $\sin 5x$ in terms of $\sin x$ and evaluate $\sin 36^\circ$.

Hint: Use sine of the sum formula and consider $\sin 5x = \sin(3x + 2x) = \sin 3x\cos 2x + \cos 3x\sin 2x$, then substituting the formulas (3.41) and (3.27) we will obtain that

$$\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x.$$

Because $\sin 180^\circ = \sin(5\cdot 36^\circ) = 0$, the above trigonometric equation can be written in terms of $x = \sin 36^\circ$ as follows, becoming a polynomial equation of fifth order that can be solved as

$$\begin{aligned}&16y^5 - 20y^3 + 5y = 0\\ &y(16y^4 - 20y^2 + 5) = 0\\ &y = 0 \quad 16y^4 - 20y^2 + 5 = 0\\ &z = y^2 \quad 16z^2 - 20z + 5 = 0\\ &z_1 = \frac{10 + \sqrt{100 - 80}}{16} = \frac{10 + 2\sqrt{5}}{16} = \frac{5 + \sqrt{5}}{8}\\ &z_2 = \frac{5 - \sqrt{5}}{8}\end{aligned}$$

$$y_1 = 0,\quad y_2 = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}, y_3 = -\frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}, y_4 = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}, y_5 = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}.$$

Though the polynomial equation has five real roots, we have to select only one answer. Our answer must be positive because 36° is in the first quadrant.

Moreover, we can state that

$$\sin 30^\circ < \sin 36^\circ < \sin 45^\circ$$
$$\frac{1}{2} < \sin 36^\circ < \frac{1}{\sqrt{2}}$$

Therefore, the only answer for $y = \sin 36^\circ = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$.

Answer: $\sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x$.

$$\sin 36^\circ = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}.$$

28. Evaluate $\sin\left(2\arctan\frac{1}{5} - \arctan\frac{5}{12}\right)$.

Answer: 0.

Solution: Denote $\arctan\frac{1}{5} = \alpha, \arctan\frac{5}{12} = \beta$, then $\tan\alpha = \frac{1}{5}$, $\tan\beta = \frac{5}{12}$. Using the formula for the tangent of a difference of two angles, we obtain

$$\tan(2\alpha - \beta) = \frac{\tan 2\alpha - \tan\beta}{1 - \tan 2\alpha \tan\beta},$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} = \frac{2\cdot\frac{1}{5}}{1 - \left(\frac{1}{5}\right)^2} = \frac{5}{12}$$

Since $\tan 2\alpha = \tan\beta = \frac{5}{12}$, then $\sin(2\alpha - \beta) = \tan(2\alpha - \beta) = 0$.

29. Prove that $\arcsin x + \arccos x = \frac{\pi}{2}$.
30. Find all values of the parameter a for which the equation has a solution: $\sin x \sin 2x \sin 3x = a$.
31. Solve the equation $\sin x + 2\sin 2x = 3 + \sin 3x$.

Answer: No solutions. Please solve it yourself using the fact that sine functions are bounded.

32. Solve the equation $\left|\cot^2 2x + 8\sqrt{-\cot 2x} - 3\right| = \left|\cot^2 2x - 8\sqrt{-\cot 2x} - 3\right|$.

Answer: $\left[\begin{array}{l} x = \frac{\pi}{4} + \frac{\pi}{2}\cdot n \\ x = \frac{5\pi}{12} + \frac{\pi}{2}\cdot l, \quad n, l \in Z. \end{array}\right.$

Solution: Denote $t = \sqrt{-\cot 2x} \geq 0, \quad \cot 2x \leq 0$.

In order not to lose any solutions we will use the following property of the absolute values:

$$|a| = |b| \Leftrightarrow |a|^2 = |b|^2 \Leftrightarrow a^2 = b^2.$$

Therefore the given equation can be rewritten in the following equivalent form:

$$\left(t^4 + (8t-3)\right)^2 = \left(t^4 - (8t+3)\right)^2$$

After squaring both sides and canceling the same terms, we obtain

$$32t^5 - 96t = 0$$

or

$$32t \cdot \left(t^4 - 3\right) = 0$$

Two cases must be considered:

Case 1 $\quad t = 0 \quad \begin{cases} \cot 2x = 0 \\ \cot 2x \le 0 \end{cases} \Leftrightarrow x = \frac{\pi}{4} + \frac{\pi}{2} \cdot n, n \in Z.$

Case 2 $\quad t^4 = 3 \Leftrightarrow t^4 = \cot^2 2x = 3$. This can be solved as follows:

$$1 + \cot^2 2x = 4$$
$$\frac{1}{\sin^2 2x} = 4$$

and then applying restrictions imposed by the square root, we have to solve the system

$$\begin{cases} \sin^2 2x = \dfrac{1}{4} \\ \cot 2x \le 0 \end{cases}$$

or to solve two different systems

$$\left[\begin{array}{l} \begin{cases} \sin 2x = \dfrac{1}{2} \\ \cot 2x \le 0 \end{cases} \Rightarrow \quad 2x = \dfrac{5\pi}{6} + 2\pi n \\ \begin{cases} \sin 2x = -\dfrac{1}{2} \\ \cot 2x \le 0 \end{cases} \Rightarrow \quad 2x = -\dfrac{\pi}{6} + 2\pi m \end{array}\right.$$

Let us explain our approach: Please place solutions to the first equation of each system on the unit circle. Each solution has two points on the unit circle. However, because cotangent is negative only in the second and fourth quadrant, we will select only such solutions that are either in the second or fourth quadrant. Further, angles $-\frac{\pi}{6} + 2\pi m$ and $\frac{5\pi}{6} + 2\pi n$ are symmetric with respect to the origin and therefore differ by π and both can be written in a simplest form as

$$2x = \frac{5\pi}{6} + \pi \cdot l, \quad l \in Z.$$

Dividing both sides by 2 we obtain

$$x = \frac{5\pi}{12} + \frac{\pi}{2} \cdot l, \quad l \in Z.$$

Uniting Case 1 and Case 2 we have the solution above.

33. Solve the equation $\sin^3 x + \cos^4 x = 1$.

Solution: Denote $y = \sin x, \quad |y| \le 1.$; then this equation will be rewritten as

$$\begin{aligned}
&y^3 + (1 - y^2)^2 = 1 \\
&y^3 + 1 - 2y^2 + y^4 = 1 \\
&y^2(y-1)(y+2) = 0 \\
&y = 0 \quad y = 1
\end{aligned}$$

1. $\begin{array}{l} \sin x = 0 \\ x = \pi n, n \in Z \end{array}$ 2. $\begin{array}{l} \sin x = 1 \\ x = \frac{\pi}{2} + 2\pi k, k \in Z \end{array}$

34. Eliminate variable t from the system $\begin{cases} x = t + 1 \\ y = t^2 - 5t - \sin t \end{cases}$.

Answer: $y = x^2 - 7x + 6 - \sin(x+1)$.

35. Solve the equation $a \sin x + b \cos x = c$ for all values of real parameters a, b, c.

Solution: Rewrite the equation in the equivalent form

$$\sqrt{a^2 + b^2} \sin(x + \varphi) = c, \quad \varphi = \arcsin \frac{b}{\sqrt{a^2 + b^2}}.$$

Then divide both sides by the coefficient of sine obtaining

$$\sin(x + \varphi) = \frac{c}{\sqrt{a^2 + b^2}}$$

1. If $c > \sqrt{a^2 + b^2}$ then there are no real solutions.
2. If $c = \pm\sqrt{a^2 + b^2}$ then

$$\begin{aligned}
&\sin(x + \varphi) = \pm 1 \\
&x = \pm\frac{\pi}{2} - \arcsin \frac{b}{\sqrt{a^2 + b^2}} + 2\pi n, \quad n \in Z.
\end{aligned}$$

3. If $|c| < \sqrt{a^2 + b^2}$ then the solution can be written as

$$x = (-1)^n \cdot \arcsin\frac{c}{\sqrt{a^2 + b^2}} - \arcsin\frac{b}{\sqrt{a^2 + b^2}} + \pi n.$$

36. Evaluate $\cos 20° \cdot \cos 40° \cdot \cos 80°$.

    Hint: Multiply and divide by $2 \sin 20°$ and then apply the sine of a double-angle formula (3.27) twice.
    Answer: 1/8.

37. Given $\tan x + \cot x = 3$. Evaluate $\tan 3x + \cot 3x$.

    Answer: 27/11.

38. Evaluate arccos(cos 10).

    Answer: $4\pi - 10$.

39. Prove that $\sin(\pi + \alpha) \cdot \sin\left(\frac{4\pi}{3} + \alpha\right) \cdot \sin\left(\frac{2\pi}{3} + \alpha\right) = \sin\left(\frac{3\alpha}{4}\right)$.

    Hint: Use formulas (3.2), (3.3), (3.41), and (3.48).

40. Solve the equation $\cos\left(2x - \frac{7\pi}{2}\right) = \sin(4x + 3\pi)$.

    Answer: $\frac{\pi}{2} \cdot n; \quad \pm\frac{\pi}{6} + \pi k, \quad n, k \in Z$.

    Hint: Use supplementary and complementary angle formulas (3.2) and (3.3), the formula for a sine of double angle (3.27), factor out a common factor, and rewrite the given equation as a product of quantities equal to zero, etc.
41. Find maximum and minimum of the function $A = xw + yz + 4x - 3w + y - 2z - 70$, if its variables satisfy the system: $\begin{cases} x^2 + y^2 - 6x - 4y - 51 = 0 \\ z^2 + w^2 + 2z + 8w - 32 = 0 \end{cases}$

    Hint: See similar Problem 139.
    Denoting $x - 3 = 8\cos\varphi$, $y - 2 = 8\sin\varphi$, $z + 1 = 7\cos\psi$, $w + 4 = 7\sin\psi$, we obtain that

$$A = (x-3)(w+4) + (z+1)(y-2) - 56$$
$$A = 56\sin(\varphi + \psi) - 56.$$

Answer: $A_{\min} = -112, \quad A_{\max} = 0$.

42. Let $0 < \alpha_1 < \ldots < \alpha_i < \ldots < \alpha_n < \frac{\pi}{2}$. Prove that the following is true:

$$\cot \alpha_1 > \frac{\cos \alpha_1 + \cos \alpha_2 + \ldots + \cos \alpha_n}{\sin \alpha_1 + \ldots + \sin \alpha_n} > \cot \alpha_n.$$

Hint: Use the fact that the cotangent function is monotonically decreasing on $\left(0, \frac{\pi}{2}\right)$ and use the ideas of Problem 141.

43. Solve the inequality: $4(1 - \tan x)^{2012} + (1 + \tan x)^{2014} \geq 2^{2014}$

Solution: Let $t = \tan x$, then the inequality

$$4(1-t)^{2012} + (1+t)^{2014} \geq 2^{2014} \tag{3.73}$$

is always valid
if $t \geq 1$ because $4(1-t)^{2012} + (1+t)^{2014} \geq (1+t)^{2014} \geq 2^{2014}$
and if $t \leq -1$ because $4(1-t)^{2012} + (1+t)^{2014} \geq (1-t)^{2012} \geq 2^{2014}$
Therefore inequality (3.73) is valid for all $|t| \geq 1$.

Consider $|t| < 1$, let $t = \cos\alpha$, $0 < \alpha < \pi$, then $4(1 - \cos\alpha)^{2012} + (1+ \cos\alpha)^{2014} = 4 \cdot \left(2\sin^2\frac{\alpha}{2}\right)^{2012} + \left(2\cos^2\frac{\alpha}{2}\right)^{2014} = 2^{2014}\left(\sin^{4024}\frac{\alpha}{2} + \cos^{4028}\frac{\alpha}{2}\right)$
$< 2^{2014}\left(\sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2}\right) = 2^{2014}$
Therefore, (3.73) is not valid for $|t| < 1$.
The original inequality has solutions is and only if $|\tan x| \geq 1$:

$$\begin{cases} \frac{\pi}{4} + \pi k \leq x \leq \frac{3\pi}{4} + \pi k \\ x \neq \frac{\pi}{2} + \pi k, k \in Z. \end{cases}$$

44. Solve the equation $x^3 - 3x - \sqrt{3} = 0$.

Hint: Because $p = -3 < 0$, $q = -\sqrt{3}$, we can use trigonometric substitution (3.51), $x = A\cos\varphi$.

Solution: $A = \sqrt{\frac{-4p}{3}} = \sqrt{\frac{-4\cdot(-3)}{3}} = 2$. The equation will be rewritten and solved as

$$8\cos^3\varphi - 6\cos\varphi - \sqrt{3} = 0$$

$$2(4\cos^3\varphi - 3\cos\varphi) = \sqrt{3}$$

$$\cos 3\varphi = \frac{\sqrt{3}}{2}$$

$$3\varphi = \arccos\left(\frac{\sqrt{3}}{2}\right) + 2\pi n$$

$$\varphi_1 = \frac{\pi}{18};\ \ \varphi_2 = \frac{13\pi}{18};\ \ \varphi_3 = \frac{21\pi}{18}$$

$$x_1 = 2\cos\frac{\pi}{18},\ \ x_2 = 2\cos\frac{13\pi}{18},\ \ x_3 = 2\cos\frac{21\pi}{18}.$$

Chapter 4
Unusual and Nonstandard Problems

When we say "nonstandard," one might think of a variety of problems that could be unusual, complex, or "impossible to solve." However, we can also say that "nonstandard" is the opposite of a standard or common way of thinking, or common methodology. For example, most of the time when we need to find the maximal or minimal value of a function, we often think of taking the derivative of a function. However, there are other techniques that can be used to solving problems about maximum and minimum values, including the use of the boundedness of functions such as trigonometric or quadratic functions, or use of different inequalities such as the inequality between arithmetic and geometric means, the Cauchy-Bunyakovsky inequality, or Bernoulli's inequality.

Another example of nonstandard problems would be problems with a parameter. We have looked at many problems with parameters in this book already but in this chapter you will see more problems with parameters and hopefully learn how to solve them. Finally, there will be some word problems that do not look standard, especially for a high school student. They are word problems to solve in integers or problems with conditions that can be reduced to a nonlinear system of equations with more variables than the number of the equations.

4.1 Problems on Maximum and Minimum

My teaching experience has been that every time students have to find a **max** or **min** of some expression or function they rush to calculate a derivative, even if this is not necessary. Sometimes the derivative approach makes finding a solution longer and more difficult.

E. Grigorieva, *Methods of Solving Nonstandard Problems*,
DOI 10.1007/978-3-319-19887-3_4

4.1.1 Using "Special" Behavior of Functions

Many problems in this section can be solved based on the properties of functions and especially the boundedness of some functions, for example, quadratic or trigonometric functions.

Problem 143 Find maximum and minimum of the function $f(x) = 4\sin x + 3\cos x$.

Solution: Applying formula (3.24) directly we obtain

$$f(x) = 4\sin x + 3\cos x = \sqrt{4^2 + 3^2}\sin(x + \varphi) = 5\sin(x + \varphi).$$

Since sine is a bounded function, Max = 5 and Min = −5.

We can see that a thorough knowledge of trigonometry allowed us to find the maximum and minimum without taking a derivative.

Problem 144 Find the maximum of the function $y(x) = 4x - x^2 + 6$.

Solution: This problem can be easily solved using the derivative but let us show an alternate way. The graph of y(x) is a parabola and it opens downward because the leading coefficient (-1) is negative. It means that the function has its maximum at its vertex:

$$x_{vertex} = -\frac{b}{2a} = \frac{-4}{2\cdot(-1)} = 2$$
$$y_{\max} = y(2) = -4 + 8 + 6 = 10.$$

Answer The max of $y(x) = 10$.

Problem 145 Find the maximum and minimum of the function $f(x) = \sin^2 x + \cos x - \frac{1}{2}.$

Solution: Using the trigonometric identity $\sin^2 x + \cos^2 x = 1$ we can rewrite the given function in the form $f(x) = 1 - \cos^2 x + \cos x - \frac{1}{2} = \frac{1}{2} - \left(\cos^2 x - \cos x\right)$

And after completing the square within parentheses we obtain

$$f(x) = \frac{3}{4} - \left(\cos x - \frac{1}{2}\right)^2$$

Notice that $\left(\cos x - \frac{1}{2}\right)^2 \geq 0$, then $f(x) \leq \frac{3}{4}$ for all real x.

On the other hand,

since $|\cos x| \leq 1$, then max of $\left(\cos x - \frac{1}{2}\right)^2 = \left(-1 - \frac{1}{2}\right)^2 = \frac{9}{4}$, at $\cos x = -1$.

And $f(x) \geq \frac{3}{4} - \frac{9}{4} = -\frac{3}{2}$ for any real x

$$-\frac{3}{2} \leq f(x) \leq \frac{3}{4} \quad \text{for} \quad \forall x \in R.$$

$$\textbf{Maximum}: \ f(x) = \frac{3}{4}, \quad \frac{3}{4} - \left(\cos\frac{\pi}{3} - \frac{1}{2}\right)^2 = \frac{3}{4}, \ x_{\max} = \frac{\pi}{3}$$

$$\textbf{Minimum}: \ f(x) = -\frac{3}{2}, \quad \frac{3}{4} - \left(\cos\pi - \frac{1}{2}\right)^2 = -\frac{3}{2}, \ x_{\min} = \pi.$$

Answer max: 3/4, min: −3/2.

Remark If you tried to solve this problem using a derivative you would encounter many troubles. First, $f(x)$ is defined for all real values of x ($x \in R$), not just on some closed interval [a, b]. Second, you can see that using boundedness and continuousness of $f(x)$ gives an accurate solution quickly.

The advantage of our approach becomes obvious and important in the next problem.

Problem 146 Find Max and Min of the function $f(x) = \log_{\frac{1}{2}}(\sin x^2 + 3)$.

Solution: Let us consider a function $f(u) = \log_{\frac{1}{2}} u$. Because the base of the logarithmic function is less than 1 ($1/2 < 1$), then $f(u)$ is decreasing over the entire domain $u > 0$. This means that for any two $\forall u_1, u_2 (u_2 > u_1 \Rightarrow f(u_2) < f(u_1))$. A sketch of such a function is given in Figure 4.1.

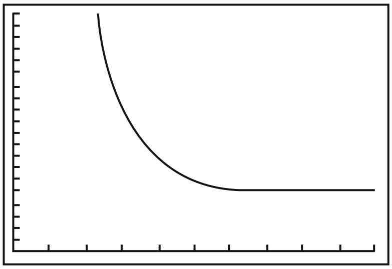


Figure 4.1 Sketch for Problem 146

In our problem $u = \sin x^2 + 3$ is bounded because $\left|\sin x^2\right| \leq 1$ and

$$u_{\min} = 1 + 3 = 4 = u_2$$
$$u_{\max} = -1 + 3 = 2 = u_1$$

Because $4 > 2$, then $f(4) < f(2)$, but since $f(x) = \log_{\frac{1}{2}}(\sin x^2 + 3)$ is monotonically decreasing then

$$\begin{aligned} \max f(x) &= f(2) = \log_{\frac{1}{2}}(2) = -1 \\ \min f(x) &= f(4) = \log_{\frac{1}{2}}(4) = -2. \end{aligned}$$

Answer Max $= -1$ and Min $= -2$.

Problem 147 Find the minimum value of $x(x+1)(x+2)(x+3)$ without calculus.

Solution: Let us multiply the middle terms together and complete the square:

$$\begin{aligned} f(x) &= x(x+3)(x+1)(x+2) = (x^2+3x)(x^2+3x+2) \\ &= \left[(x^2+3x)^2 + 2\cdot(x^2+3x) + 1\right] - 1 \\ &= ((x^2+3x)+1)^2 - 1 \geq -1. \end{aligned}$$

We can see that -1 is the minimum value of our function.

Answer Min $f(x) = -1$.

Problem 148 Prove that for any $x > 0$ the inequality $x^2 + \pi x + \dfrac{15}{2}\pi \sin x > 0$ is true.

Proof Because $|\sin x| \leq 1$, then the minimum of $\sin x = -1$, and the left side can be written as $x^2 + \pi x - 7.5\pi$, where $x = \dfrac{3\pi}{2} + 2\pi n, n = 0, 1, 2, 3\ldots (x > 0)$. It is obvious that when $x = \dfrac{3\pi}{2}$ $(n = 0)$ the left side approaches its possible minimum. So we have to show that $\left(\dfrac{3\pi}{2}\right)^2 + \pi\cdot\dfrac{3\pi}{2} - \dfrac{15\pi}{2} > 0.$

Multiplying both sides by 4 we get $9\pi^2 + 6\pi^2 - 30\pi = 15\pi(\pi - 2) > 0$ because $\pi > 2$. We have proven that the given inequality is greater than 0 for any $x > 0$.

Problem 149 Find the minimum of the function
$$f(x) = \frac{\sin\left(x - \frac{\pi}{6}\right)}{4\sin\left(x - \frac{\pi}{6}\right)\cos 3x - \cos 6x - 7}$$

Solution:
Method 1: First, we will not use any derivative for solving this problem:

$$f(x) = \frac{\sin\left(x - \frac{\pi}{6}\right)}{4\sin\left(x - \frac{\pi}{6}\right)\cos 3x - \cos 6x - 7}$$

Let us substitute $\cos 6x = 2\cos^2 3x - 1$ and simplify the given function as

$$f(x) = \frac{\sin\left(x - \frac{\pi}{6}\right)}{4\sin\left(x - \frac{\pi}{6}\right)\cos 3x - 2\cos^2 3x - 6}$$
$$= -\frac{\sin\left(x - \frac{\pi}{6}\right)}{2\left(\cos^2 3x - 2\sin\left(x - \frac{\pi}{6}\right)\cos 3x + 3\right)}$$

We can define a new variable $y = \cos 3x$ and rewrite the denominator. Since the quadratic portion of the denominator, $y^2 - 2\sin\left(x - \frac{\pi}{6}\right)y + 3 > 0$ (because the discriminant will always be negative, since) $\left|\sin\left(x - \frac{\pi}{6}\right)\right| \leq 1$, then our goal is to find the minimum possible value of the denominator. Completing the square inside the denominator, we obtain

$$f(x) = -\frac{\sin\left(x - \frac{\pi}{6}\right)}{2\left(\left(\cos 3x - \sin\left(x - \frac{\pi}{6}\right)\right)^2 + 3 - \sin^2\left(x - \frac{\pi}{6}\right)\right)} = -\frac{a}{2\left[(y-a)^2 + 3 - a^2\right]};$$
$$a = \sin\left(x - \frac{\pi}{6}\right);\ y = \cos 3x.$$

We can see that the minimal value of the function, $\min f(x) = -\frac{1}{4}$, will occur if $a = 1$, $y = a$. The following must hold:

$$\begin{cases} \cos 3x - \sin\left(x - \frac{\pi}{6}\right) = 0 \\ \sin\left(x - \frac{\pi}{6}\right) = 1 \end{cases} \Rightarrow x = \frac{2\pi}{3} + 2\pi n.$$

Therefore $\min f(x) = -\frac{1}{4}$.

Method 2: 1. Denote $x-\frac{\pi}{6}=\alpha \Rightarrow x=\alpha+\frac{\pi}{6}$, and then

$$\begin{aligned}
3x &= 3\alpha+\frac{\pi}{2} \Rightarrow \cos 3x = \cos\left(3\alpha+\frac{\pi}{2}\right) = -\sin 3\alpha\\
6x &= 6\alpha+\pi \Rightarrow \cos 6x = \cos(6\alpha+\pi) = -\cos 6\alpha
\end{aligned}$$

In terms of this new variable our original function will be rewritten as

$$f(\alpha)=\frac{\sin(\alpha)}{-4\sin(\alpha)\sin 3\alpha+\cos 6\alpha-7}=\frac{\sin(\alpha)}{-4\sin(\alpha)\sin 3\alpha+1-2\sin^2 3\alpha-7}$$

$$f(\alpha)=\frac{-\sin(\alpha)}{2(\sin^2 3\alpha+2\sin 3\alpha\sin\alpha+3)}$$

In order to simplify "the look" of the function we will introduce new variables: $a=\sin\alpha, b=\sin 3\alpha$

$$F(a,b)=-\frac{a}{2\left((b+a)^2+3-a^2\right)} \to \min \tag{4.1}$$

Let us look closely at function (4.1).

First, we can write the following true inequalities:

$$\begin{gathered}
(b+a)^2+3-a^2 \geq 3-a^2 \quad (|a|\leq 1)\\
\frac{1}{2\left((b+a)^2+3-a^2\right)} \leq \frac{1}{2(3-a^2)}
\end{gathered} \tag{4.2}$$

We can state that (4.1) is true if the following is true:

$$\frac{a}{2\left((b+a)^2+3-a^2\right)} \to \max \tag{4.3}$$

If we put together (4.3) and (4.2), then we can write the chain of true inequalities:

$$F(a,b)=\frac{-a}{2\left((b+a)^2+3-a^2\right)} \geq \frac{-a}{2(3-a^2)} \tag{4.4}$$

Second, the maximum of $\frac{a}{2(3-a^2)}$ occurs at $a=1$ and equals ¼.

Finally,

$$\min(f(x))=\min(F(a,b))=-\frac{1}{4}. \tag{4.5}$$

Let us find for what values of x the statement (4.5) will hold.

The equality in (4.4) holds if

$$\begin{cases} a = 1 \\ b = a \end{cases} \Leftrightarrow \begin{cases} \sin\alpha = 1 \\ \sin 3\alpha = 1 \end{cases} \Leftrightarrow \sin\left(x - \frac{\pi}{6}\right) = 1 \Rightarrow x = \frac{2\pi}{3} + 2\pi n, n \in Z.$$

Answer $\mathrm{Min}(f(x)) = -\frac{1}{4}.$

Problem 150 There are three alloys. The first alloy contains 30 % of Pb and 70 % of Sn. The second—80 % of Sn and 20 % of Zn, and the 3rd—50 % of Sn, 10 % of Pb, and 40 % of Zn. From these three alloys we need to prepare a new alloy containing 15 % of lead (Pb). What is the maximum and minimum percentage of Sn that can be in these alloys?

Solution: Assuming that we are melting x kg of the 1st alloy, y kg of the 2nd alloy and z kg of the 3rd, let us create the following table:

Because the new alloy must contain 15 % of Pb we can obtain the equation

$$\frac{0.3x + 0.1z}{x + y + z} = 0.15 \tag{4.6}$$

Multiplying both sides of (4.6) by $100(x+y+z)$ we obtain

$$\begin{aligned} &30x + 10z = 15x + 15y + 15z \\ &15x = 15y + 5z \\ &x = y + \frac{z}{3} \end{aligned} \tag{4.7}$$

(4.7) expresses x in terms of y and z.

We have to find the maximum and minimum possible content of Sn in the alloy:

$$F(x, y, z) = F = \frac{0.7x + 0.8y + 0.5z}{x + y + z} \tag{4.8}$$

Using (4.7) let us exclude variable x from (4.8):

$$F = \frac{0.7\left(y + \frac{z}{3}\right) + 0.8y + 0.5z}{\left(y + \frac{z}{3}\right) + y + z} = \frac{1.5y + \frac{2.2z}{3}}{\left(y + \frac{z}{3}\right) + y + z} = \frac{1.5y + \frac{2.2z}{3}}{2y + \frac{4z}{3}}$$

Table 4.1 Problem 150

# of alloy	Amount (kg)	Sn (%)	Pb (%)	Zn (%)
1	X	70	30	0
2	Y	80	0	20
3	Z	50	10	40

Multiplying the numerator and denominator by 30, we obtain that

$$F = \frac{45y + 22z}{60y + 40z} \tag{4.9}$$

The amount of Sn in the alloy is a function of two variables y and z ($y > 0$, $z > 0$). Equation (4.9) can be written in the form

$$F = \frac{45y + 30z - 8z}{60y + 40z} = \frac{3(15y + 10z) - 8z}{4(15y + 10z)}$$
$$F = \frac{3}{4} - \frac{2z}{15y + 10z}$$
$$F = \frac{3}{4} - g(y, z)$$

Because $z \geq 0$, $y \geq 0$ let us make $z = \text{constant} = z_0$ and consider F as a function of y, $y \geq 0$ for $z = z_0, F(y) = \frac{3}{4} - g(y)$, where $g(y) > 0$ and $F(y)$ decreases for all y. This means that the minimum of $g(y)$ will correspond to the maximum of $F(y)$:

$$Min = F(0) = \frac{3}{4} - \frac{2\cancel{z}}{10\cancel{z}} = 0.55 \text{ or } 55\%$$

We notice that the minimum does not depend on z.
The maximum of F can be reached only if $z = 0$.
Max of $F = 3/4$ or 75 %.

Actually $F(z = 0) = \frac{3}{4} - \frac{2 \cdot 0}{15y + 10 \cdot 0} = \frac{3}{4}$.

Notice that $g(y, z) > 0$ if $y > 0$ and $z > 0$ and $g(y, z) = 0$ if $y > 0$ but $z = 0$.

Answer 55 % and 75 %.

4.1.2 Using Arithmetic and Geometric Mean

Many problems about the maximum and minimum values of a function can be solved without using a derivative, but instead using the inequality between the arithmetic and geometric means:

$$\frac{a + b}{2} \geq \sqrt{ab} \text{ for } a > 0 \text{ and } b > 0 \tag{4.10}$$

Inequality (4.10) becomes an equality only when $a = b$. For any other a and b an arithmetic mean (AM) is always greater than the geometric mean (GM). This can be proved as follows:

$$(a - b)^2 \geq 0$$
$$a^2 - 2ab + b^2 \geq 0$$
$$a^2 + 2ab + b^2 \geq 4ab$$
$$(a + b)^2 \geq 4ab$$
$$a + b \geq 2\sqrt{ab}$$

In 1893 the Russian collector Golenishchev purchased an Egyptian papyrus, which was about eighteen feet long and about three inches high. From a sample of problems from the papyrus it was clear that the inequality between arithmetic and geometric means was known to ancient Egyptians in 1850 BC. Egyptians were impressively good at building pyramids and they used a geometric approach for establishing important relationships. They also introduced the so-called harmonic mean (HM) and knew that

$$\mathrm{HM} \leq \mathrm{GM} \leq \mathrm{AM}.$$

Let us demonstrate that (4.10) is true using plane geometry.

We will construct a circle with diameter $(a+b)$. Let $a = BD$, $b = DC$, and A be the point where the perpendicular to BC at point D intersects the circle and let E be the foot of the perpendicular from D to the radius AO. Let us denote $AD = h$, $AE = g$. Since ABD and CAD are similar right triangles then

$$\frac{h}{b} = \frac{a}{h} \Rightarrow h = \sqrt{ab}.$$

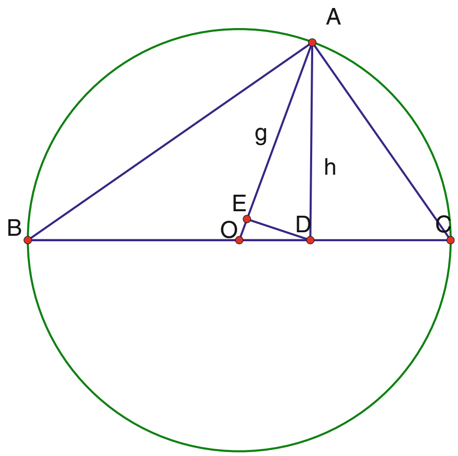
Also, since AOD and ADE are similar right triangles, we have

$$\frac{g}{\sqrt{ab}} = \frac{\sqrt{ab}}{\frac{a+b}{2}} \Rightarrow g = \frac{2ab}{a+b} = \frac{2}{\left(\frac{1}{a}+\frac{1}{b}\right)}$$

Finally, from geometry we know that in a right triangle, the length of any leg is always smaller than the length of the hypotenuse (Figure 4.2). Hence, $AE \leq AD \leq AO \Leftrightarrow g \leq h \leq \frac{a+b}{2}$, which can be rewritten as

$$\frac{2ab}{a+b} \leq \sqrt{ab} \leq \frac{a+b}{2} \tag{4.11}$$

Figure 4.2 Arithmetic and geometric means on a circle



Problem 151 For all positive x, y, and z find the minimum value of $(x+y)(y+z)$ if $xyz(x+y+z)=9$.

Solution: Since $xyz(x+y+z)=9$ can be written as

$$xz\left(yx+y^2+yz\right)=9. \tag{4.12}$$

we can represent $(x+y)(y+z)$ as

$$(y+x)(y+z)=y(y+z)+x(y+z)=\left(y^2+yz+xy\right)+xz.$$

Substituting for the term inside the parentheses from (4.12) and using the inequality between arithmetic and geometric mean we obtain

$$(x+y)\,(y+z)=\frac{9}{xz}+xz\geq 2\sqrt{\frac{9}{xz}\cdot xz}=2\cdot 3=6$$

Therefore, 6 is the minimum value of $(x+y)(y+z)$.

Answer $\mathrm{Min}=6$.

Remark A similar problem (Problem 172) is solved in Section 4.3 using a geometric approach.

Problem 152 Two vessels contain different solutions of salt. There is 5 kg of solution in the first vessel and 20 kg in the second. After evaporation the percentage of salt in the first vessel increased p times, and in the second q times. It is known that $p\times q=9$. What is the maximum amount of water that would have evaporated from both vessels together?

Solution: Let us introduce two variables:

x (kg) is the amount of water that evaporated from the first vessel.

y (kg) is the amount of water that evaporated from the second.

Translating the problem into the language of math we obtain the system:

$$\begin{cases} 5=p(5-x)\\ 20=q(20-y)\\ p\cdot q=9\\ p>1\\ q>1\end{cases}\Leftrightarrow\begin{cases} x=5-\dfrac{5}{p}\\ y=20-\dfrac{20}{q}\\ pq=9\\ 1<p<q\end{cases}\Leftrightarrow\begin{cases} x=5-\dfrac{5}{p}\\ y=20-\dfrac{20p}{9}\\ pq=9\\ 1<p<q\end{cases}$$

Using this system, let us find the maximum of $(x+y)$ and consider

$$x+y=5-\frac{5}{p}+20-\frac{20p}{9}=25-\left(\frac{5}{p}+\frac{20p}{9}\right) \tag{4.13}$$

Of course, we could take the derivative of $(x+y)$ with respect to p or solve it graphically on a calculator. But using the inequality between arithmetic and geometric means and the fact that $1 \le p \le 9$ (4.13) can be written as

$$x+y=f(p)=25-\left(\frac{5}{p}+\frac{20p}{9}\right) \le 25-2\sqrt{\frac{5}{p}\cdot\frac{20p}{9}}=25-\frac{20}{3}=18\frac{1}{3} \tag{4.14}$$

From (4.14) we conclude that the max $\underset{1\le p\le 9}{f(p)} = 18\frac{1}{3}$.

Let's find the value of p that makes $f(p)$ approach the maximum:
$\frac{5}{p}+\frac{20p}{9}=2\sqrt{\frac{5}{p}\cdot\frac{20p}{9}}$ if and only if $\frac{5}{p}=\frac{20p}{9}$ or $p^2=\frac{9}{4}$, $p=\frac{3}{2}$:

$$\max f(p)=f\left(\frac{3}{2}\right)=18\frac{1}{3}$$
$$q=\frac{9}{p}=6.$$

Answer The maximum amount of water that can have evaporated from the two vessels is $18\frac{1}{3}$ kg. Moreover, the salt content in the 1st vessel increased 1.5 times and in the 2nd vessel 6 times.

Remark This problem is also interesting because we were not given specific concentrations of the solutions in the 1st and the 2nd vessels, but we were able to solve the problem without that information. If you do not understand how the first system was obtained let's look at an example. Let us assume that the 1st solution is 20 % salt, and the second 50 % salt. Thus 5 kg of the first solution contains 0.2×5 kg of the salt, and 20 kg of the 2nd solution will contain 0.5×20 kg of the salt. After x kg of water is evaporated from the 1st solution, $(5-x)$ kg of solution contains p times as much concentration of the salt as before, meaning $0.2 \times p$. This gives us

$$0.2\cdot 5=0.2\cdot p\cdot(5-x)$$
$$5=p(5-x)$$

(See the first equation of the 1st system.)

Problem 153 Find the minimum value of the function $f(x)=4x+\frac{9\pi^2}{x}+\sin x$ for $x>0$.

Solution: Suggestion 1: If you try to use a derivative you will get a transcendental equation:

$$f'(x) = 4 - \frac{9\pi^2}{x^2} + \cos x = 0.$$

It is hard to solve this equation analytically.

Suggestion 2: Try to look at $f(x)$ as a sum of two functions $g(x)$ and $h(x)$ such that $g(x) = 4x + \frac{9\pi^2}{x}$ and $h(x) = \sin x$. For all $x > 0$, the function $g(x)$ has a lower bound because $4x + \frac{9\pi^2}{x} \geq 2\sqrt{4x \cdot \frac{9\pi^2}{x}} = 12\pi$ and this happens at $4x = \frac{9\pi^2}{x}$ or $x = \frac{3\pi}{2}$.

Is it a good number?

$h(x) = \sin x$ is a periodic function, and $|\sin x| \leq 1$ for all real x. Then $f(x)$ must have its minimum (Min) at

$$x = \frac{3\pi}{2} \approx 4.71$$
$$\text{min of } f(x) = 12\pi - 1 \approx 36.7.$$

Answer $f_{\min}(x) = 12\pi - 1.$

Remark Of course, some of you would say that using technology, one could get the answer in 2 s. Let us do it on a TI-84 graphing calculator. The function appeared in window: $0 < x < 70$ and $-5 < y < 100$. The function is continuous for all $x > 0$ and using **[2nd calc]** we can find the minimum $x_{\min} \approx 4.7, y_{\min} \approx 36.7.$ This is very close to the exact answer. Just think of how easily we obtained the exact answer analytically and what excellent techniques we have learned.

4.1.3 Using Other Important Inequalities

Let us unite our knowledge about methods of estimation and evaluation and list three very famous inequalities.

The Inequality of Cauchy
If

$$\begin{aligned} &a_1 \geq 0, a_2 \geq 0, \ldots, a_n \geq 0 \Rightarrow \\ &\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdot \ldots \cdot a_n} \end{aligned} \tag{4.15}$$

This inequality becomes an equality if and only if $a_1 = a_2 = \ldots = a_n$.

We very often call on this when we use the inequality between geometric and arithmetic means.

Bernoulli's Inequality

This inequality can be written as:

For any $x > -1$ and any natural number n we can state that

$$(1 + x)^n \geq 1 + nx \ (\mathrm{B.1}) \tag{4.16}$$

This becomes an equality only if $x = 0$ and $n = 1$.

Next, let us prove Bernoulli's inequality (Theorem 22) using mathematical induction. To ensure that it is an inequality we will use the condition that $x \neq 0$ and $n \geq 2$.

Theorem 22 *If* $x > -1, x \neq 0, n \geq 2$ *then* $(1 + x)^n \geq 1 + nx$.

Proof Proof by induction

1. If $n = 2$, then $(1 + x)^2 = 1 + 2x + x^2 > 1 + 2x$, which is obviously true.
2. Let $n = k$ and assume that $A(k)$ is true and that

$$(1 + x)^k > 1 + kx$$

3. Let us prove that $A(k + 1)$ is also true, i.e., $(1 + x)^{k+1} > 1 + (k + 1)x$.
   Consider

$$(1 + x)^{k+1} = (1 + x)^k (1 + x) > (1 + kx)(1 + x) = 1 + (k + 1)x + kx^2 > \ 1 + (k + 1)x$$

The proof is completed.

Besides (B.1, (4.16)) there exists a more general Bernoulli's inequality, which contains two inequalities:

If $p < 0$ or $p > 1$, then

$$(1 + x)^p \geq 1 + px \tag{4.17}$$

If $0 < p < 1$, and $x > -1$, then

$$(1 + x)^p \leq 1 + px \tag{4.18}$$

The Inequality of Cauchy-Bunyakovsky (CB)

$$(x_1 y_1 + x_2 y_2 + \ldots + x_n y_n)^2 \leq \left(x_1^2 + x_2^2 + \ldots + x_n^2\right)\left(y_1^2 + y_2^2 + \ldots + y_n^2\right) \tag{4.19}$$

Inequality **(CB)** becomes an equality if and only if $x_k = a y_k$, where $a > 0$ and $k = 1, 2, 3, \ldots n$.

Let us demonstrate how this inequality can be proven with the use of the scalar product of two vectors. Without loss of generality (WLOG), we will consider two

two-dimensional vectors $x = (x_1, x_2), y = (y_1, y_2)$. It is well known that the scalar product of two vectors equals the product of their magnitudes times the cosine of the angle between two vectors, which can be written as

$$(x, y) = x \cdot y = \|x\| \cdot \|y\| \cos \alpha.$$

Next, we will square both sides and will consider the fact the cosine is a bounded function less than or equal to one. Thus, the proof is below:

$$\begin{aligned}
&(x \cdot y)^2 = \|x\|^2 \cdot \|y\|^2 \cos^2\alpha \\
&(x_1 \cdot y_1 + x_2 \cdot y_2)^2 = \left(x_1^2 + x_2^2\right) \cdot \left(y_1^2 + y_2^2\right) \cos^2\alpha \\
&(x_1 \cdot y_1 + x_2 \cdot y_2)^2 \le \left(x_1^2 + x_2^2\right) \cdot \left(y_1^2 + y_2^2\right)
\end{aligned}$$

The CB inequality can be applied to solving many interesting problems.

I offer you several problems below that can be solved using these famous inequalities. Try to recognize which is the appropriate one to use in each case.

Problem 154 Which is greater $100\,!$ or 10^{200}?

Solution: It is not possible to do it on a typical graphing calculator: the numbers are too big. However, we can use Cauchy's inequality. It is obvious that

$$\begin{aligned}
&\frac{1+2+3+\ldots+100}{100} \ge \sqrt[100]{1 \cdot 2 \cdot 3 \cdot \ldots \cdot 100} = \sqrt[100]{100!} \\
&50.5^{100} \ge 100! \\
&10^{200} = 100^{100} > 50.5^{100} > 100!
\end{aligned}$$

Answer $10^{200} > 100!$

Problem 155 Which is greater $200\,!$ or 100^{200}?

Solution: Let us consider the following ratio:

$$\frac{200!}{100^{200}} = \left(\frac{1}{100} \cdot \frac{199}{100}\right)\left(\frac{2}{100} \cdot \frac{198}{100}\right)\left(\frac{3}{100} \cdot \frac{197}{100}\right)\cdots$$
$$\cdots\left(\frac{99}{100} \cdot \frac{101}{100}\right)\left(\frac{100}{100} \cdot \frac{200}{100}\right) =$$

The numerator of each fraction inside parentheses can be rewritten using the difference of squares formula:

$$\frac{200!}{100^{200}} = \left(\frac{(100-99)}{100}\cdot\frac{(100+99)}{100}\right)\left(\frac{(100-98)}{100}\cdot\frac{(100+98)}{100}\right)\cdots$$
$$\cdots\left(\frac{(100-1)}{100}\cdot\frac{(100+1)}{100}\right)(1\cdot 2) = \tag{4.20}$$
$$= \frac{100^2-99^2}{100^2}\cdot\frac{100^2-98^2}{100^2}\cdots\frac{100^2-1^2}{100^2}\cdot 1\cdot 2$$

Since all fractions in (4.20) are less than 1, then $200! < 100^{200}$.

Problem 156 Given $a+b+c=1$. Prove that $a^2+b^2+c^2 \geq \frac{1}{3}$.

Proof 1 Consider two vectors, $x=(a,b,c)$ and $y=(1,1,1)$. From **CB** inequality we have

$$(a\cdot 1+b\cdot 1+c\cdot 1)^2 \leq \left(a^2+b^2+c^2\right)\cdot\left(1^2+1^2+1^2\right)$$

After substituting $a+b=c=1$ and taking the square root of both sides, we have

$$1 = 1\times a+1\times b+1\times c \leq \sqrt{1^2+1^2+1^2}\cdot\sqrt{a^2+b^2+c^2}$$
$$= \sqrt{3}\cdot\sqrt{a^2+b^2+c^2}$$

From this we obtain that $a^2+b^2+c^2 \geq \frac{1}{3}$.

Proof 2 On one hand, because $a+b+c=1$ then $(a+b+c)^2=1$ and

$$1=(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$$

On the other hand, using the inequality between geometric and arithmetic means, we know that

$$2ab \leq a^2+b^2, \quad 2ac \leq a^2+c^2, \quad 2bc \leq b^2+c^2.$$

Thus, $1 \leq 3a^2+3b^2+3c^2 = 3\left(a^2+b^2+c^2\right)$

Then $a^2+b^2+c^2 \geq \frac{1}{3}$.

The proof is completed.

Problem 157 Prove that $a^4 + b^4 \geq \frac{1}{8}$ for any a and b such that $a > 0$, $b > 0$ $a + b = 1$.

Proof Let

$$a = \frac{1}{2} - \alpha, \quad \alpha > 0$$
$$b = \frac{1}{2} + \alpha$$

then $a + b = 1$ is true and

$$a^4 + b^4 = \frac{1}{8} + 3\alpha^2 + 2\alpha^4 \geq \frac{1}{8} \quad \forall \alpha.$$

Problem 158 Prove that if $a > 0$ and $b > 0$ and $a^3 + b^3 = a^5 + b^5$, then $a^2 + b^2 \leq 1 + ab$.

Proof Let us proof it by contradiction. Suppose that

$$a^2 + b^2 > 1 + ab \tag{4.21}$$

Separately multiplying (4.21) by a^3 and b^3 we obtain two inequalities:

$$a^3\left(a^2 + b^2\right) > a^3(1 + ab)$$
$$b^3\left(a^2 + b^2\right) > b^3(1 + ab)$$

Adding these two inequalities we obtain:

$$a^5 + a^3b^2 + a^2b^3 + b^5 > a^3 + a^4b + b^3 + ab^4$$

Because by the condition of the problem $a^3 + b^3 = a^5 + b^5$, then

$$a^3b^2 + a^2b^3 > a^4b + ab^4 \Rightarrow a^2b + ab^2 > a^3 + b^3$$

Factoring the left side of the last inequality and applying the sum of cubes formula to the right side, and dividing both of the sides by $(a + b) > 0$ we have

$$ab > a^2 - ab + b^2$$
$$(a - b)^2 < 0$$

We understand that this inequality is false, then $a^2 + b^2 \leq 1 + ab$.

Problem 159 Find the maximum value of $f(x) = \sqrt{1 - \frac{x}{3}} + \sqrt[6]{1 + x}$.

Solution: Let us apply Bernoulli's inequality to both terms on the right and then add the left and right sides:

$$\left.\begin{array}{l}\left(1 - \dfrac{x}{3}\right)^{\frac{1}{2}} \leq 1 - \dfrac{1}{2} \cdot \dfrac{x}{3} \\ (1 + x)^{\frac{1}{6}} \leq 1 + \dfrac{1}{6} \cdot x\end{array}\right\} \Rightarrow f(x) \leq 2.$$

Therefore the maximum value of the function is 2 at $x = 0$.

Answer Max of $f(x)$ is 2 at $x = 0$.

Problem 160 (Suprun) Find maximum and minimum of the function $f(x, y) = 6 \sin x \cos y + 2 \sin x \sin y + 3 \cos x$.

Solution: Cauchy-Bunyakovsky inequality (4.19) applied to three-component vectors can be written as $(x_1 y_1 + x_2 y_2 + x_3 y_3)^2 \leq \left(x_1^2 + x_2^2 + x_3^2\right) \cdot \left(y_1^2 + y_2^2 + y_3^2\right)$ and for our problem:

$$\begin{aligned}&(f(x, y))^2 \leq \left(6^2 + 2^2 + 3^2\right)\left((\sin x \cos y)^2 + (\sin x \sin y)^2 + \cos^2 x\right) \\ &f^2(x, y) \leq 49(\sin^2 x[\cos^2 y + \sin^2 y] + \cos^2 x) = 49 \\ &|f(x, y)| \leq 7\end{aligned}$$

Therefore the minimum value of the function is -7 and maximum is 7.

Answer Max $= 7$, Min $= -7$.

Problem 161 For the polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$ with real coefficients a, b, c, and d and with four real zeros, x_1, x_2, x_3, x_4, the following relationship is valid: $b - d \geq 5$. Find the minimal value of the product $A = \left(x_1^2 + 1\right)\left(x_2^2 + 1\right)\left(x_3^2 + 1\right)\left(x_4^2 + 1\right) \to \min$.

Solution: (One of several methods). As we did in Problem 90, the polynomial with four real zeros can be written as a product of two quadratic functions:

$$p(x) = x^4 + ax^3 + bx^2 + cx + d = \left(x^2 + px + q\right)\left(x^2 + rx + s\right) \tag{4.22}$$

And the value of A in terms of p, q, r, s was obtained earlier as

$$A = \left((q-1)^2 + p^2\right)\left((s-1)^2 + r^2\right) \tag{4.23}$$

Comparing coefficients of the original polynomial and its factorized form (4.22), we have that

$$b = q + s + pr; \quad d = qs$$

Therefore, the given inequality is

$$\begin{aligned} &b - d \geq 5 \\ &q + s + pr - qs \geq 5 \\ &pr \geq qs - q - s + 1 + 4 \end{aligned}$$

The last inequality can be written as

$$pr \geq (q-1)(s-1) + 4$$

or as

$$pr - (q-1)(s-1) \geq 4 \tag{4.24}$$

Next, we will prove the following statement:

$$(x^2 + y^2)(z^2 + t^2) \geq (yt - xz)^2, \quad x, y, z, t \in R \tag{4.25}$$

Proof Expanding both sides we have

$$\begin{aligned} &x^2z^2 + x^2t^2 + y^2z^2 + y^2t^2 \geq y^2t^2 - 2xyzt + x^2z^2 \\ &x^2t^2 + y^2z^2 \geq -2xyzt \\ &(xt + yz)^2 \geq 0 \quad \text{true.} \end{aligned}$$

Therefore, the inequality (4.25) is true.

Note that the validity of a similar inequality $(x^2 + y^2)(z^2 + t^2) \geq (yt + xz)^2$ follows from the Cauchy-Bunyakovsky inequality.

Using (4.25) and (4.24), we can rewrite (4.23) as follows:

$$A = \left((q-1)^2 + p^2\right)\left((s-1)^2 + r^2\right) \geq (pr - (q-1)(s-1))^2 \geq 4^2 = 16.$$

Therefore the smallest value of A is 16. In order to verify this case we can consider a polynomial: $p(x) = (x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$.

Answer Min = 16.

Problem 162 Given $x_1 > 0, x_2 > 0, x_3 > 0$. Prove that
$\dfrac{x_1}{x_2+x_3} + \dfrac{x_2}{x_1+x_3} + \dfrac{x_3}{x_1+x_2} \geq \dfrac{3}{2}$.

Proof Let $x_2 + x_3 = a,\;\; x_3 + x_1 = b,\;\; x_1 + x_2 = c \;\Rightarrow$

$$2(x_1 + x_2 + x_3) = a + b + c$$
$$x_1 + a = \frac{a+b+c}{2}$$
$$x_1 = \frac{b+c-a}{2}$$
$$x_2 = \frac{a+c-b}{2}$$
$$x_3 = \frac{a+b-c}{2}$$

The left side of the inequality is rewritten as

$$\frac{b+c-a}{2a} + \frac{a+c-b}{2b} + \frac{a+b-c}{2c}$$

Next, within each fraction, we will divide each term of the numerator by the denominator and regroup the fractions, combine reciprocals, and apply Cauchy's inequality to each pair:

$$\frac{1}{2}\left[\frac{b}{a} + \frac{c}{a} - 1 + \frac{a}{b} + \frac{c}{b} - 1 + \frac{a}{c} + \frac{b}{c} - 1\right] =$$
$$\frac{1}{2}\cdot\left[\underbrace{\left(\frac{a}{b} + \frac{b}{a}\right)}_{\geq 2}\right] + \frac{1}{2}\cdot\left[\underbrace{\left(\frac{a}{c} + \frac{c}{a}\right)}_{\geq 2}\right] + \frac{1}{2}\cdot\left[\underbrace{\left(\frac{c}{b} + \frac{b}{c}\right)}_{\geq 2}\right] - \frac{3}{2} \geq 3 - \frac{3}{2} = \frac{3}{2}$$

The proof is completed.

Problem 163 Given $x, y, z \in R$ satisfying the system $\begin{cases} x+y+z=5 \\ xy+xz+yz=8 \end{cases}$

Prove that all the variables are bounded as $1 \le x \le \frac{7}{3}$, $1 \le y \le \frac{7}{3}, 1 \le z \le \frac{7}{3}$.

Proof If we square the first equation of the system and make a corresponding substitution, we will obtain that

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xy$$
$$5^2 = x^2 + y^2 + z^2 + 2 \cdot 8$$
$$x^2 + y^2 + z^2 = 9$$

Additionally, we can rewrite the system as

$$\begin{cases} y+z = 5-x \\ yz + x(y+z) = 8 \end{cases} \Rightarrow$$
$$yz = 8 - x(5-x) = x^2 - 5x + 8$$

We rewrite the last equality as

$$yz = x^2 - 5x + 8$$

On the other hand, using Cauchy's inequality, we can state that

$$y + z \ge 2\sqrt{yz}$$
$$5 - x \ge 2\sqrt{yz}$$
$$(5-x)^2 \ge 4yz$$

Since the last inequality can be rewritten as

$$4yz \le (5-x)^2,$$

Finally, we obtain that

$$4(x^2 - 5x + 8) \le 25 + x^2 - 10x$$
$$3x^2 - 10x + 7 \le 0$$
$$1 \le x \le \frac{7}{3}.$$

Similarly, we can prove the two other inequalities.

4.2 Nonstandard Systems and Equations in Two or Three Variables

I remember how at an educational conference one high school math teacher said to me: "Why do I need all these "tricks" if I can solve everything on a calculator?" Our dispute was over about five minutes after I asked her to tell me which was greater: 2^{300000} or 3^{200000}. Technology is a great thing, it can be very helpful, and certainly it enhances math teaching, but it must be used wisely. Regarding this section, first, let us discuss what type of systems can be solved using a typical graphing calculator (not a computer). There are systems of k linear equations in k variables with nonsingular coefficient matrices (with nonzero determinants) and nonlinear systems in two variables like

$$\begin{cases} 3x^3 - y + 10 = 0 \\ y - 2x + 2^{x^4} - 5\sin 3x = 4 \end{cases}$$

Each equation of such a system can be written as an explicit function of x:

$$\begin{cases} y = 3x^3 + 10 \\ y = 2x - 2^{x^4} + 5\sin 3x + 4 \end{cases}$$

And we would solve such a system graphically by looking for the intersections of the two graphs. What can you do if a system is nonlinear, no variable can be expressed as an explicit function of another, or a system has more than two variables? In this section we will learn some methods of solving nonstandard systems and equations in two or three variables. 1978 Lomonosov Moscow State University admission exam had the following problem.

Problem 164 Find all ordered pairs (x, y) that for all nonnegative y satisfy the system of equations: $\begin{cases} 4^{x^2+(y+1)^2} - 32 = 31 \cdot 2^{x^2+(y+1)^2} \\ \cos\left[\pi(x^2 + y^2)\right] = 1 \end{cases}$

Solution: Let us rewrite the system, adding an inequality to it:

$$\begin{cases} 4^{x^2+(y+1)^2} - 32 = 31 \cdot 2^{x^2+(y+1)^2} \\ \cos\left[\pi(x^2 + y^2)\right] = 1 \\ y \geq 0 \end{cases} \tag{4.26}$$

Solve equations of the system separately:

$$4^{x^2+(y+1)^2} - 32 - 31 \cdot 2^{x^2+(y+1)^2} = 0$$

Let

$$2^{x^2+(y+1)^2} = z \tag{4.27}$$

We obtain a quadratic equation:

$$\begin{aligned} &z^2 - 31z - 32 = 0 \\ &z_1 = -1, \text{ and } z_2 = 32 \end{aligned}$$

But we choose only $z = 32$ because $z > 0$ since it is an exponential function. Then

$$\begin{aligned} &2^{x^2+(y+1)^2} = 32 \\ &x^2 + (y+1)^2 = 5 \\ &x^2 + y^2 + 2y = 4 \\ &\cos\left(\pi(x^2 + y^2)\right) = 1 \end{aligned} \tag{4.28}$$

From the last equation we have

$$\begin{aligned} &\pi(x^2 + y^2) = 2\pi n \\ &x^2 + y^2 = 2n, \quad n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Now we can rewrite a new system combining (4.27), (4.28), and the first equation of (4.26):

$$\begin{cases} x^2 + y^2 + 2y = 4 \\ x^2 + y^2 = 2n \geq 0 \\ y \geq 0 \end{cases} \tag{4.29}$$

Subtracting the second equation of system (4.29) from the first, we obtain

$$\begin{cases} y = 2 - n \\ x^2 + y^2 = 2n \\ y \geq 0 \end{cases} \Rightarrow \begin{cases} 0 \leq n \leq 2 \\ x^2 + (2-n)^2 = 2n \end{cases} \Leftrightarrow \begin{cases} 0 \leq n \leq 2 \\ x^2 = 6n - n^2 - 4 \end{cases} \tag{4.30}$$

It is easy to see from (4.30) that $n = 0$, 1, or 2.

Let us investigate for which value of n system (4.26) will have solutions? To do this we attempt to solve system (4.30) for $n = 0$, $n = 1$, and $n = 2$.

1. $\begin{cases} n = 0 \\ x^2 = 6n - n^2 - 4 \\ y = 2 - n \end{cases} \Leftrightarrow \begin{cases} x^2 = -4 \\ y = 2 \\ n = 0 \end{cases} \Leftrightarrow \varnothing$ system has no solutions
2. $\begin{cases} n = 1 \\ x^2 = 6n - n^2 - 4 \\ y = 2 - n \end{cases} \Leftrightarrow \begin{cases} n = 1 \\ x = \pm 1 \\ y = 1 \end{cases} \Leftrightarrow (1, 1)$ and $(-1, 1)$ solutions

3. $\begin{cases} n = 2 \\ x^2 = 6n - n^2 - 4 \\ y = 2 - n \end{cases} \Leftrightarrow \begin{cases} n = 2 \\ x^2 = 4 \\ y = 0 \end{cases} \Leftrightarrow \left[\begin{array}{l} x = \pm 2 \\ y = 0 \end{array} \right.$

Answer (1,1), (−1,1), (−2,0), and (2,0)

I hope that you can see the advantage of having the ability to solve such systems by hand—you develop your math skills and it would be impossible to solve such a system on a calculator.

Problem 165 Solve the system:

$$\begin{cases} 2(y-2)(y-z) = z - 2 \\ 4x^2 + z^2 = 4z \\ 8x^3 - z = 3xy \\ z \le 2 \end{cases} \tag{4.31}$$

Solution: This system is nonstandard. Each equation of (4.31) is an implicit function of two or three variables x, y, and z. In addition, one of the equations contains the variable $\boldsymbol{x}$ raised to the third degree. Using a graphing calculator would be difficult, as would an attempt to find triples (x, y, z) mentally. Let us try the technique of solving this system using boundedness of some variables. Just a quick look at the second equation of the system gives us the idea that since $4x^2 + z^2 \ge 0$ always, this equation has solutions only if its right side is nonnegative as well. Now $z \ge 0$. Using the inequality condition of (4.31) we obtain that

$$0 \le z \le 2 \tag{4.32}$$

Let us solve the first equation of the system separately:

$$2\left(y^2 - zy - 2y + 2z\right) = 2y^2 - 2zy - 4y + 4z = z - 2.$$

Considering this equation as a quadratic equation in one variable $\boldsymbol{y}$, and $\boldsymbol{z}$ as a parameter, we can try to solve it for y:

$$\begin{aligned} &2y^2 - 2(z+2)y + (3z+2) = 0 \\ &y = \frac{z + 2 \pm \sqrt{z(z-2)}}{2} \end{aligned} \tag{4.33}$$

Notice that (4.33) has real solutions for y if and only if $z(z-2) \ge 0$ (its discriminant is nonnegative). Thus,

$$z \le 0 \quad \text{or} \quad z \ge 2 \tag{4.34}$$

If we solve (4.32) and (4.34) together we obtain

$$z = 0 \quad \text{or} \quad z = 2$$

Using some "nontrivial thinking" we have found that solutions of system (4.31) may exist only if $z = 0$ or $z = 2$.

Case 1 $z = 0$

Replacing $z = 0$ in the system we obtain

$$\begin{cases} 2(y-2)(y-0) = 0-2 \\ 4x^2 = 0 \\ 8x^3 = 3xy \\ z = 0 \end{cases} \Leftrightarrow \left[\begin{array}{l} y = 1 \\ x = 0 \\ z = 0 \end{array}\right. \Leftrightarrow (0,1,0)$$

Case 2 $z = 2$

Replacing $z = 2$ in the system, we obtain

$$\begin{cases} 2(y-2)(y-2) = 2-2 \\ 4x^2 + 2^2 = 4 \cdot 2 \\ 8x^3 - 2 = 3xy \\ z = 2 \end{cases} \Leftrightarrow \left[\begin{array}{l} y = 2 \\ x = \pm 1 \\ 8x^3 - 6x - 2 = 0 \\ z = 2 \end{array}\right. \Leftrightarrow (1,2,2)$$

We successfully found two triples.

Answer (0, 1, 0) and (1, 2, 2).

Problem 168 Find all solutions of the system:

$$\begin{cases} y + 2 = (3-x)^3 \\ (2z - y)(y+2) = 9 + 4y \\ x^2 + z^2 = 4x \\ z \geq 0 \end{cases}$$

Solution: Let us first work with the second equation and divide it by $y + 2 \neq 0$. This allows us to express variable z as follows:

$$z = \frac{1}{2}\left(y + 2 + \frac{1}{y+2} + 2\right) \tag{4.35}$$

Because it is nonnegative by the condition of the problem $(z \geq 0)$, we can rewrite the relationship above as an inequality:

$$y + 2 + \frac{1}{y+2} \geq -2$$

Additionally we can see that this variable is bounded because it is the sum of two reciprocals. Then the inequality above is equivalent to two cases.

Case 1 $y + 2 = -1, y = -3, x = 4, z = 0.$

Case 2 $y + 2 > 0$. Then by the property of two positive reciprocal numbers we obtain that $y + 2 + \frac{1}{y+2} \geq 2$. Then it follows from (4.35) that $z \geq 2$.

Next, let us complete the square in the third equation of the system:

$$x^2 + z^2 = 4x$$
$$(x-2)^2 + z^2 = 4$$

Now we can see that in order for this equation to be true, variable z must be less than or equal to 2: $(z \leq 2)$. Therefore, $z = 2, x = 2, y = -1$.

Answer {(4,–3,0);(2.–1,2)}.

Problem 167 Solve the system of the inequalities:

$$\begin{cases} \left|\sin \dfrac{\pi(x+y)}{2}\right| + (x-y-2)^2 \leq 0 \\ |2x+3| \leq 2 \end{cases}.$$

Solution: At first glance this system looks hard but if we look at the first inequality more closely we notice that absolute values are nonnegative and $(x-y-2)^2 \geq 0$. However, the sum of two nonnegative expressions must be nonpositive (the right side of the inequality). This can happen only if

$$\sin \frac{\pi(x+y)}{2} = 0 \tag{4.36}$$

and

$$(x-y-2)^2 = 0 \tag{4.37}$$

simultaneously!

Then we get $0+0 \le 0$ which is true enough. Let us solve (4.36) and (4.37) separately:

$$\sin\frac{\pi(x+y)}{2} = 0$$
$$\frac{\pi(x+y)}{2} = \pi \cdot n, \quad n = 0, \pm 1, \pm 2, \ldots$$
$$x+y = 2n$$
$$x-y-2=0$$
$$x-y=2$$

From the second inequality of the given system we can obtain more restrictions on variable $\boldsymbol{x}$:

$|2x+3| \le 2$ is equivalent to a double inequality

$$-2 \le 2x+3 \le 2 \quad \text{or} \quad -5 \le 2x \le -1 \tag{4.38}$$

Combining (4.36)–(4.38) we obtain the system:

$$\begin{cases} x-y=2 \\ x+y=2n \\ -5 \le 2x \le -1 \end{cases} \tag{4.39}$$

Adding the first two equations will give us

$$\begin{cases} 2x = 2n+2 \\ -5 \le 2x \le -1 \end{cases}$$

Excluding x from the system we have

$$-7 \le 2n \le -3 \quad \text{or} \quad -3.5 \le n \le -1.5$$

But because $\boldsymbol{n}$ is a whole number there are just two possible numbers n that satisfy this inequality. Those are $n=-3$ and $n=-2$. Now for each value of n we can solve the system (4.39):

1. $\begin{cases} n=-3 \\ x=n+1 \\ y=x-2 \end{cases} \Leftrightarrow (-2, -4)$
2. $\begin{cases} n=-2 \\ x=n+1 \\ y=x-2 \end{cases} \Leftrightarrow (-1, -3)$

Answer We found two ordered pairs: $(-2,-4)$ and $(-1,-3)$.

Problem 168 Find all triples (x, y, z) such that
$\sqrt{-2x^2 - 2y^2 + 2z^2 - 11x\sqrt{6} + 2y + 6z} + \sqrt{2\sqrt{6}x^2 + 3\sqrt{6} - 12x\cos\pi y\cos\pi z} = 0.$

Solution: The first idea that should come to your mind is that $\underset{\geq 0}{\sqrt{\ }} + \underset{\geq 0}{\sqrt{\ }} = 0$ has any sense if and only if both radicands are equal to zero simultaneously:

$$-2x^2 - 2y^2 + 2z^2 - 11x\sqrt{6} + 2y + 6z = 0 \tag{4.40}$$

$$2\sqrt{6}x^2 + 3\sqrt{6} - 12x\cos\pi y\cos\pi z = 0 \tag{4.41}$$

Let us solve the second equation rewriting a product of cosines as a sum:

$$2\sqrt{6}x^2 + 3\sqrt{6} - 6x[\cos\pi(y-z) + \cos\pi(y+z)] = 0 \tag{4.42}$$

Denoting

$$a = \cos\pi(y-z) + \cos\pi(y+z) \tag{4.43}$$

We rewrite (4.42) as a quadratic type:

$$2x^2 - a\sqrt{6}\cdot x + 3 = 0 \tag{4.44}$$

Knowing that a quadratic equation has real roots only if its discriminant (D) is greater or equal to zero, find D:

$$D = 6a^2 - 24 \geq 0 \Leftrightarrow a^2 - 4 \geq 0 \Leftrightarrow a \leq -2 \ \text{ or } \ a \geq 2 \tag{4.45}$$

Look at the expression (4.41) again. Because of the boundedness of the *cosine* function

$$-2 \leq a \leq 2 \tag{4.46}$$

(4.45) and (4.46) together will give us two possible situations for a:

$$\left[\begin{array}{l} a = \cos\pi(y-z) + \cos\pi(y+z) = 2 \\ a = \cos\pi(y-z) + \cos\pi(y+z) = -2 \end{array}\right.$$

This system can be split into the union of two systems:

1.
$$\left[\begin{cases} \cos\pi(y-z) = 1 \\ \cos\pi(y+z) = 1 \end{cases}\right. \Leftrightarrow \begin{cases} \pi(y-z) = 2\pi n \\ \pi(y+z) = 2\pi l \end{cases} \Leftrightarrow \begin{cases} y - z = 2n \\ y + z = 2l \end{cases} \tag{4.47}$$

From (4.47) we conclude that if $a = 2$, then either both variables (y, z) are even numbers or both are odd.

2.
$$\left[\begin{cases}\cos\pi(y-z) = -1\\ \cos\pi(y+z) = -1\end{cases}\right. \Leftrightarrow \begin{cases}\pi(y-z) = \pi + 2\pi n\\ \pi(y+z) = \pi + 2\pi l\end{cases} \Leftrightarrow \begin{cases}y-z = 1+2n\\ y+z = 1+2l\end{cases} \tag{4.48}$$

From (4.48) we see that since for $a = -2$ the difference and the sum of y and z are an odd, only such ordered pairs (y, z) are possible: (odd, even) or (even, odd).

Let us return us to (4.44) again. If $D = 0$, then $a = 2$ or $a = -2$ and quadratic (4.44) can have two solutions:

$$x_1 = \frac{\sqrt{6}}{2} \text{ and } x_2 = -\frac{\sqrt{6}}{2}$$

Let us look at (4.40) after replacing x by $\frac{\sqrt{6}}{2}$:

$$\begin{aligned}&-2\cdot\frac{6}{4} - 2y^2 + 2z^2 - 11\cdot\frac{\sqrt{6}}{2}\cdot\sqrt{6} + 2y + 6z = 0\\ &y^2 - z^2 - y - 3z + 18 = 0\end{aligned}$$

Let us rewrite this equation as $y^2 - y = z^2 + 3z - 18$ and then factor the left and the right side of it

$$y(y-1) = (z+6)(z-3)$$

Since variables y and z are integers, the following pairs (y, z) satisfy the equation above:

$$(y,z) : \{(0,3), (1,3), (0,-6), (1,-6)\}$$

However, only the first three ordered pairs, (0, 3), (1, 3), and (0, −6), satisfy (4.47) (their difference and sum are even). Combining them with $x = \frac{\sqrt{6}}{2}$ we obtain three triples:

$$(x,y,z) : \left\{\left(\frac{\sqrt{6}}{2}, 0, 3\right), \left(\frac{\sqrt{6}}{2}, 1, 3\right), \left(\frac{\sqrt{6}}{2}, 0, -6\right)\right\} \tag{4.49}$$

Note that since for $a = -2\left(x = -\frac{\sqrt{6}}{2}\right)$ (4.40) does not have integer solutions for y and z, (4.49) will be the answer.

Answer $(x,y,z) : \left\{\left(\frac{\sqrt{6}}{2}, 0, 3\right), \left(\frac{\sqrt{6}}{2}, 1, 3\right), \left(\frac{\sqrt{6}}{2}, 0, -6\right)\right\}.$

4.3 Geometric Approach to Solving Algebraic Problems

Sometimes it is beneficial to look at an algebraic problem from a geometric point of view. For example, for positive values of x and y, the equation $x^2 + y^2 = a^2$ can be seen as the relationship between sides of a right triangle with hypotenuse a and legs x and y. Next, think about the following equation:

$$x^2 + xy + y^2 = a^2. \tag{4.50}$$

Can this also be considered as a relationship between sides of a triangle? Please recall the law of cosines for a triangle with sides $a, b, c : a^2 = b^2 + c^2 - 2bc \cdot \cos(\angle A)$, where $\angle A$ is opposite of side a. We can see that if $\angle A = 120°$, then $a^2 = b^2 + c^2 + bc$. Thus, if sides x and y form an angle of $120°$, then (4.50) expresses the third side, a, of a triangle in terms of x and y.

Next let us look at the following system of nonlinear equations:

$$\begin{cases} y^2 + yz + z^2 = b^2 \\ x^2 + xz + z^2 = c^2 \\ x^2 + xy + y^2 = a^2 \end{cases}$$

Imagine now that we have a triangle ABC with sides a, b, and c. Also imagine a point P inside the triangle connected with each of the three vertices of triangle ABC (see Figure 4.3) such that the following is true:

$$\angle APB = \angle BPC = \angle CPA = 120°.$$

If we now need to evaluate $A = xy + xz + zy$, for positive values of x, y, and z that satisfy the system above. Let us show that each term of A can be expressed in terms of the area of one of the three small triangles in which big triangle ABC is divided by the interior point P (see Figure 4.3).

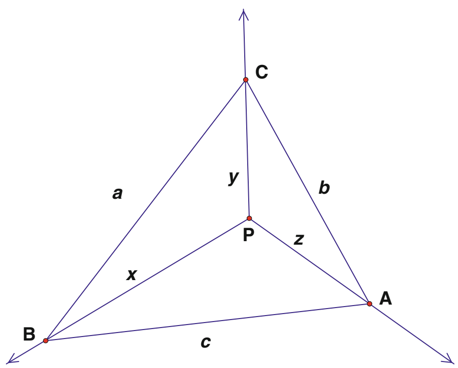


Figure 4.3 Geometric view of formula (4.50)

For example, expressing the area of each small triangle as half of the product of two sides and the sine of the angle between them and with the help of Figure 4.3, we can state the following:

$$[BPC] = \frac{1}{2} \cdot x \cdot y \cdot \sin 120^\circ = xy\frac{\sqrt{3}}{4}$$
$$[BPA] = \frac{1}{2} \cdot x \cdot z \cdot \sin 120^\circ = xz\frac{\sqrt{3}}{4}$$
$$[CPA] = \frac{1}{2} \cdot z \cdot y \cdot \sin 120^\circ = zy\frac{\sqrt{3}}{4}$$

Adding the left and right sides we will get the area of the big triangle that can be found using, for example, Heron's formula:

$$\frac{\sqrt{3}}{4}(xy + xz + zy) = \sqrt{p(p-a)(p-b)(p-c)},\ p = \frac{a+b+c}{2}.$$

This formula can be solved for A:

$$\frac{A\sqrt{3}}{4} = \frac{\sqrt{(b+c-a)(a+b-c)(a+c-b)(a+b+c)}}{4}$$
$$A = \sqrt{\frac{(b+c-a)(a+b-c)(a+c-b)(a+b+c)}{3}}.$$

Next, I want us to look at the following problem.

Problem 169 Prove that for any positive a, b, and c the following is always true: $\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} \geq \sqrt{a^2 + ac + c^2}$. For what values of the parameters does this inequality become an equality?

Proof From point M let us draw three rays such that $\angle AMB = 120^\circ$, $\angle AMC = 60^\circ$, $\angle CMB = 60^\circ$, where points A, C, and B are such that $AM = a$, $MC = b$, $MB = c$ (see Figure 4.4).

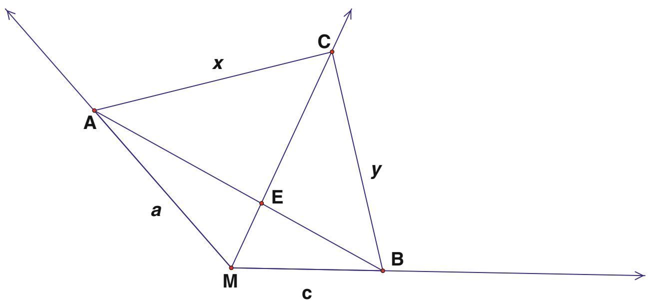


Figure 4.4 First sketch for Problem 169

Connect A and C, C and B, and A and B, respectively. Denote $AC = x$ and $CB = y$. Applying law of cosines to triangles AMC and CMB, respectively, we have the following true relationships:

$$\begin{cases} x = \sqrt{a^2 - 2ab\cos 60^\circ + b^2} \\ y = \sqrt{b^2 - 2bc\cos 60^\circ + c^2} \end{cases}$$ which can also be written as

$$x = \sqrt{a^2 - ab + b^2}$$
$$y = \sqrt{b^2 - bc + c^2}$$

Now for side AB in triangle AMB we obtain

$$|AB| = \sqrt{a^2 + ac + c^2}.$$

Finally, because $x = AC$, $y = CB$, and AB are the sides of triangle ACB, then their lengths must satisfy the triangle inequality (the sum of two sides is greater than the third side): $AC + CB > AB$, which algebraically satisfies the strict inequality:

$$\sqrt{a^2 - ab + b^2} + \sqrt{b^2 - bc + c^2} > \sqrt{a^2 + ac + c^2}.$$

If point C belongs to side AB, then $AC + CB = AB$ or

$x + y = AB$ and the inequality becomes the equality. Let us find the relationship between parameters a, b, and c.

Consider a new triangle describing this case in Figure 4.5. Because MC is an angle bisector, then applying the Triangle Bisector Theorem (see for example, my book "Methods of Solving Complex Geometry Problems," page 45 [3]) we obtain

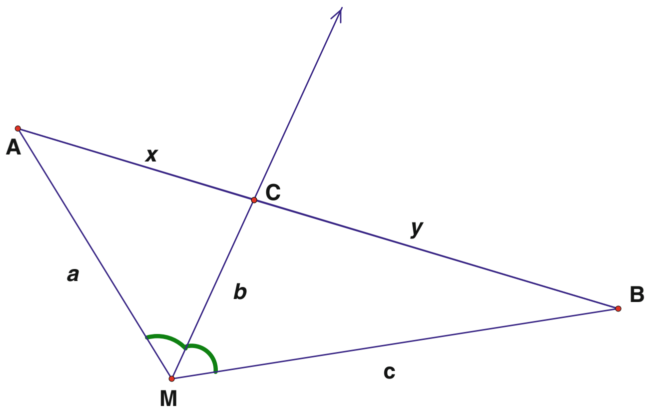


Figure 4.5 Final sketch for Problem 169

$$\frac{AM}{AC} = \frac{MB}{CB}$$
$$\frac{a}{x} = \frac{c}{y}$$

From the last equation and after substituting for x and y their expressions using law of cosines, we obtain

$$\left(\frac{x}{y}\right)^2 = \left(\frac{a}{c}\right)^2$$
$$\frac{a^2}{c^2} = \frac{a^2 - ab + b^2}{b^2 - bc + c^2}$$
$$a^2\left(b^2 - bc + c^2\right) = c^2\left(a^2 - ab + b^2\right)$$
$$a^2b^2 - a^2bc = c^2b^2 - c^2ab$$

We can divide all sides by the common factor $b \neq 0$ and then simplify as follows:

$$a^2b - a^2c = c^2b - c^2a$$
$$b(a^2 - c^2) = ac(a - c)$$
$$b = \frac{ac}{a + c}$$

This can also be written as $\frac{1}{b} = \frac{1}{a} + \frac{1}{c}$.

The proof is completed.

Problem 170 Evaluate $A = xy + 2yz + 3xz$, if the positive numbers x, y, and z satisfy the system: $\begin{cases} 3x^2 + 3xy + y^2 = 75 \\ y^2 + 3z^2 = 48 \\ x^2 + xz + z^2 = 9 \end{cases}$.

Solution: This system can be rewritten as

$$\begin{cases} x^2 - 2x \cdot \frac{y}{\sqrt{3}} \cos 150^\circ + \left(\frac{y}{\sqrt{3}}\right)^2 = 25 & \text{(Law of Cosines)} \\ \left(\frac{y}{\sqrt{3}}\right)^2 + z^2 = 16 & \text{(Pythagorean Theorem)} \\ x^2 - 2xz \cos 120^\circ + z^2 = 9 & \text{(Law of Cosines)} \end{cases}$$

This new system now can be seen geometrically as a relationship between the sides of a geometric figure (Figure 4.6). For example, consider a triangle ABC and a point P as its interior point:

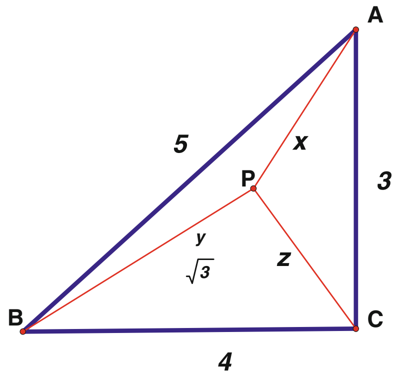


Figure 4.6 Sketch for Problem 170

$$x = AP, \quad z = PC, \quad \frac{y}{\sqrt{3}} = BP, \quad \angle APC = 120^\circ, \quad \angle APB = 150^\circ, \quad \angle BPC = 90^\circ.$$

The area of triangle ABC equals the sum of the areas of the three inside triangles, APB, APC, and BPC:

$$[APB] = \frac{1}{2} \cdot x \cdot \frac{y}{\sqrt{3}} \sin 150^\circ = \frac{xy}{4\sqrt{3}}$$
$$[BPC] = \frac{1}{2} \cdot z \cdot \frac{y}{\sqrt{3}} = \frac{2yz}{4\sqrt{3}}$$
$$[APC] = \frac{1}{2} \cdot x \cdot z \sin 120^\circ = \frac{3xz}{4\sqrt{3}}$$

If we add the left sides and the right sides, respectively, and using the fact that $[ABC] = \frac{1}{2} \cdot 4 \cdot 3 = 6$, we obtain a new formula:

$$6 = \frac{1}{4\sqrt{3}} (xy + 2yz + 3xz)$$
$$A = xy + 2yz + 3xz = 24\sqrt{3}.$$

Answer $24\sqrt{3}$.

Problem 171 Given positive numbers x, y, and z such that $xyz(x + y + z) = 1$. Find the minimum of $(x + y) \cdot (z + x)$.

Solution: 1. Let us imagine a triangle ABC with sides a, b, and c. Denote its half perimeter as $p = \dfrac{a + b + c}{2}$.

Next we can introduce positive variables x, y, and z as

$$x = p - a = \frac{b + c - a}{2}$$
$$y = p - b = \frac{a + c - b}{2}$$
$$z = p - c = \frac{a + b - c}{2}$$

From the above we obtain the following true relationships:

$$\left.\begin{array}{r} x + y = c \\ x + z = b \end{array}\right\} \Rightarrow (x + y)(x + z) = c \cdot b$$

$$\left.\begin{array}{l} xyz = (p - a)(p - b)(p - c) \\ x + y + z = 3p - 2p = p \end{array}\right\} \Rightarrow$$
$$xyz(x + y + z) = p \cdot (p - a)(p - b)(p - c) = 1.$$

2. On the other hand, using Heron's formula for the area of triangle ABC, we can state the following:

$$[ABC] = \sqrt{p(p - a)(p - b)(p - c)} = 1.$$

3. Expressing the same unit area using half of the product of two sides and the sine of the angle between them we obtain

$$[ABC] = \frac{1}{2} c \cdot b \cdot \sin \angle A = 1.$$

This can be solved for the product of two sides:

$$cb = \frac{2}{\sin \angle A}.$$

4. Finally, we have

$$(x + y)(x + z) = \frac{2}{\sin \angle A}.$$

Because the maximum of sine is one, then the minimum value of the quantity is two.

Answer Min of $(x + y)\,(x + z) = 2$.

4.4 Trigonometric Substitution

In some problems it is appropriate to make a trigonometric substitution in order to solve a much easier equation or a system of equations. For example let us consider the following problem.

Problem 172

Solve the system $\begin{cases} x^2 + y^2 = 1 \\ 4xy(2y^2 - 1) = -1 \end{cases}$.

Solution: This system can be solved by expressing x from the second equation in terms of y and substituting it into the first equation. However, it will result in a polynomial equation of the 6th order, the solution of which would be problematic (please try).

If we look instead at the first equation, we will notice that the ordered pairs (x,y) must belong to the unit circle. Hence the following trigonometric substitution seems to be reasonable:

$$x = \cos t, \quad y = \sin t, \quad 0 \le t < 2\pi$$

The first equation is just a Pythagorean identity and the second equation becomes

$$\begin{aligned} &4\cos t \sin t(2\sin^2 t - 1) = -1 \\ &2\sin 2t(\cancel{1} - \cos 2t - \cancel{1}) = -1 \\ &\sin 4t = 1 \\ &4t = \frac{\pi}{2} + 2\pi n \\ &\begin{cases} t = \dfrac{\pi}{8} + \dfrac{\pi}{2} \cdot n \\ 0 \le t \le 2\pi \end{cases} \Rightarrow n = 0, 1, 2, 3 \end{aligned}$$

Finally, we get the answer for x and y below.

Answer $x = \cos\left(\frac{\pi}{8} + \frac{\pi}{2}n\right), \quad y = \sin\left(\frac{\pi}{8} + \frac{\pi}{2}n\right), \quad n = 0, 1, 2, 3$.

In the following problem, we will have to find maximum value of the quantity $x + y$.

Problem 173 Find the maximum value of the expression $A = x + z$ if x, y, z, w satisfy the following system $\begin{cases} x^2 + y^2 = 4 \\ z^2 + t^2 = 9 \\ xt + yz = 6 \end{cases}$.

Solution: We can see that $(x,\ y)$ and $(z,\ t)$ belong to the circles of radius 2 and 3, respectively. Using a substitution:

$$\begin{aligned} x &= 2\cos\varphi, y = 2\sin\varphi \\ z &= 3\cos\psi, t = 3\sin\psi, \quad \varphi, \psi \in [0, 2\pi) \end{aligned}$$

The last equation of the system will be rewritten as

$$\begin{aligned} &6\cos\varphi\sin\psi + 6\sin\varphi\cos\psi = 6 \\ &\sin(\varphi + \psi) = 1 \\ &\varphi = \frac{\pi}{2} - \psi \Rightarrow \\ &\cos\varphi = \sin\psi, \sin\varphi = \cos\psi \end{aligned}$$

Hence,

$$\begin{aligned} A &= 2\cos\varphi + 3\cos\psi \\ A &= 2\cos\varphi + 3\sin\varphi = \sqrt{13}\sin(\varphi + \alpha) \end{aligned}$$

It follows from the boundedness of the sine function that

$$A_{\max} = (x+z)_{\max} = \sqrt{13}.$$

Problem 174 (MGU 2004 exam) Find maximum and minimum values of the expression $\dfrac{y^2}{121} + \dfrac{w^2}{81}$ if x, y, z, w satisfy the following system:

$$\begin{cases} x^2 + y^2 - 8x + 6y - 96 = 0 \\ z^2 + w^2 + 10z - 4w - 52 = 0 \\ xw + yz - 2x - 4w + 5y + 3z - 76 \geq 0 \end{cases}$$

Solution: Completing the square we can rewrite the first two equations as

$$\begin{aligned} (x-4)^2 + (y+3)^2 &= 121 \\ (z+5)^2 + (w-2)^2 &= 81 \end{aligned}$$

The last inequality can be transformed as

$$\begin{aligned} &(w-2)x + (z+5)y - 4w + 3z \geq 76 \\ &(w-2)(x-4) + 4(w-2) + ((z+5)(y+3)) - 3(z+5) - 4w + 3z \geq 76 \\ &(w-2)(x-4) + (z+5)(y+3) \geq 76 + 8 + 15 = 99 \end{aligned}$$

Using trigonometric substitutions, such as the following

$$\begin{cases} x-4=11\cos\varphi \\ y+3=11\sin\varphi \\ z+5=9\cos\psi \\ w-2=9\sin\psi \end{cases}$$

we can rewrite the last inequality of the given system as

$$9\sin\psi\cdot 11\cos\varphi+9\cos\psi\cdot 11\sin\varphi\geq 99.$$

After simplification and applying the formula for the sine of the sum of two angles, we obtain

$$\sin(\varphi+\psi)\geq 1.$$

Because sine is a bounded function, the inequality above must only be the equality $\sin(\varphi+\psi)=1$ with the solution

$$\psi=\frac{\pi}{2}-\varphi+2\pi n.$$

Hence, using the property of complementary angles we can state the following true relationship:

$$\begin{aligned} \sin\psi &= \cos\varphi \\ \cos\psi &= \sin\varphi \end{aligned}$$

Next, we need to find maximum and minimum of

$$\begin{aligned} \frac{y^2}{121}+\frac{w^2}{81} &= \frac{(11\sin\varphi-3)^2}{121}+\frac{(2+9\cos\varphi)^2}{81} \\ &= 1+\frac{9}{121}+\frac{4}{81}+\left(-\frac{6}{11}\sin\varphi+\frac{4}{9}\cos\varphi\right) \\ &= \frac{11014}{9801}+\frac{2\sqrt{1213}}{99}\cdot\sin(\alpha-\varphi) \end{aligned}$$

Using boundedness of the sine function we finally get the answer.

Answer Max/Min $=\dfrac{11014}{9801}\pm\dfrac{2\sqrt{1213}}{99}$.

Problem 175 Solve the system: $\begin{cases} 2x+x^2y=y \\ 2y+y^2z=z \\ 2z+z^2x=x \end{cases}$.

Solution: First, let us rewrite the system in a different form:

$$\begin{cases} y = \dfrac{2x}{1-x^2} \\ z = \dfrac{2y}{1-y^2} \\ x = \dfrac{2z}{1-z^2} \end{cases}$$

If we use a trigonometric substitution for x:

$$x = \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

Then we can apply the formula for the tangent of a double angle.

Recall that $\tan 2t = \dfrac{2\tan t}{1-\tan^2 t}$.

Applying this formula to all of the equations, we will get the following:

$$\begin{aligned} y &= \tan 2t \\ z &= \tan 4t \\ x &= \tan 8t \\ \tan t &= \tan 8t \end{aligned}$$

The last equation has a solution:

$$\begin{aligned} &\tan 8t - \tan t = 0 \\ &\tan 7t = 0 \\ &7t = \pi n \\ &\begin{cases} t = \dfrac{\pi}{7} \cdot n \\ -\dfrac{\pi}{2} < t < \dfrac{\pi}{2} \end{cases} \\ &-\frac{\pi}{2} < \frac{\pi}{7} \cdot n < \frac{\pi}{2} \\ &-\frac{7}{2} < n < \frac{7}{2} \end{aligned}$$

Therefore, $(x, y, z) = \left(\tan \dfrac{\pi n}{7}, \tan \dfrac{2\pi n}{7}, \tan \dfrac{4\pi n}{7}\right), \quad -3 \le n \le 3.$

Answer $(x, y, z) = \left(\tan \dfrac{\pi n}{7}, \tan \dfrac{2\pi n}{7}, \tan \dfrac{4\pi n}{7}\right), \quad -3 \le n \le 3.$

4.5 Problems with Parameters

In this book, we have already considered some equations and systems with parameters. I like problems with parameters. They are not just challenging but also very useful because they usually represent a real situation. Both variables can depend on a parameter (thickness of a wall, acidity of solution, etc.). If we have an equation with one variable, say, x and with one parameter, a, then we can in general write it as $F(x, a) = 0$. Usually (unless additional restrictions are applied) we assume that parameter a represents all real numbers, and so we need to solve the given equation for all possible values of a parameter. For demonstration we can start from a quadratic equation with a parameter, for example, with the following problem:

Solve the equation $x^2 + 3x + a = 0$ for all real values of a parameter a.

Since we know that the number of the solutions will depend on the discriminant, we can evaluate it as $D = 3^2 - 4a = 9 - 4a$, and then if the discriminant is positive $\left(a < \frac{9}{4}\right)$, the equation will have two distinct roots, if $a = 9/4$, then only one root, and if the discriminant is negative, then for $a > 9/4$ the quadratic function will have no zeros and the graph of the parabola will never intersect the X-axis.

Problems with a parameter can be very challenging and sometimes the methods for solving them do not come easily. For example, what would you do, if I gave you the following problem:

Find all values of parameter a, such that $\log_5(a\cos 2x - (1 + a^2 - \cos^2 x)\sin x + 4 - a) \leq 1$ is true for any real x.

Ideas for solving the problem above and for many others are well explained in this section. Though solutions of some of the problems with a parameter can be predicted on a graphing calculator (and we will discuss it here), only an algebraic approach gives you an opportunity to see the physical aspect of the situation, to be a real participant of the situation, and not just a "button-presser." Moreover, no calculators are allowed on mathematics Olympiads anymore, so you need to learn how to do problems analytically; I will help you to succeed.

Problem 176 Real x, y, and a satisfy the system: $\begin{cases} x + y = a - 1 \\ xy = a^2 - 7a + 14 \end{cases}$.
Find for what values of a does the sum $x^2 + y^2$ approach a maximum.

Solution: I am sure we could find both x and y in terms of parameter a, then add their squares ... But this way seems to be very long and complicated. Try it!

Let us rewrite $x^2 + y^2$ differently. The fact of the matter is that we don't need x and y themselves; we need $x^2 + y^2$. Our system gives us $(x+y)$ and xy "directly." Our task is now combining the first and second equations of the system to express $x^2 + y^2$ in terms of a.

Recalling that $(x+y)^2 = x^2 + 2xy + y^2$, then

$$x^2 + y^2 = (x+y)^2 - 2xy.$$

The expression above tells us to square both sides of the first equation of system, to multiply both sides of the second equation by 2, and to separately subtract the left and the right sides of our new equations:

$$-\begin{cases} (x+y)^2 = a^2 - 2a + 1 \\ 2xy = 2a^2 - 14a + 28 \end{cases}$$
$$x^2 + y^2 = -a^2 + 12a - 27 \tag{4.51}$$

Completing the square on the right side of (4.51) we obtain

$$x^2 + y^2 = -(a-6)^2 + 9 \tag{4.52}$$

From which we see that when $a = 6$, $x^2 + y^2$ approaches its maximum value, $x^2 + y^2 = 9$. Note that for any other real a different from $a = 6$, the sum $(x^2 + y^2)$ given by (4.52) is less than 9.

Answer $a = 6$.

Problem 177 Find all values of parameter a, such that $\log_5(a\cos 2x - (1 + a^2 - \cos^2)\sin x + 4 - a) \le 1$ is true for any real x.

Solution:

$$\log_5(a\cos 2x - (1 + a^2 - \cos^2)\sin x + 4 - a) \le 1$$
$$0 < a(1 - 2\sin^2 x) - (\sin^2 x + a^2)\sin x + 4 - a \le 5$$
$$-5 \le \sin^3 x + 2a\sin^2 x + a^2 \sin x - 4 < 0$$
$$-1 \le \sin x(\sin x + a)^2 < 4 \tag{4.53}$$

If the inequality (4.53) is true for any $x \in R$, then it must be true for x such that $\sin x = \pm 1$. Now, (4.53) can be written as

$$\begin{cases} -1 \le 1(1+a)^2 < 4 \\ -1 \le (-1)(-1+a)^2 < 4 \end{cases} \Rightarrow \begin{cases} -2 \le 1 + a < 2 \\ -1 \le -1 + a \le 1 \end{cases} \Leftrightarrow 0 \le a < 1.$$

Let us show that for all values of parameter a from [0, 1) (4.53) is true $\forall x \in R$. We will show separately that

$$\sin x(\sin x + a)^2 < 4 \quad \text{and} \quad \sin x(\sin x + a)^2 \geq -1$$

Using the boundedness of $\sin x$ we obtain

$$-1 \leq \sin x + a < 2 \Rightarrow \sin x(\sin x + a)^2 < 1 \cdot 2^2 = 4 \tag{4.54}$$

If $\sin x \geq 0$, then $\sin x(\sin x + a)^2 \geq 0 \geq -1$.

If $\sin x < 0$, then $-1 \leq \sin x(\sin x + a)^2 < 1$.

Therefore,

$$\sin x(\sin x + a)^2 \geq -1. \tag{4.55}$$

Inequalities (4.54) and (4.55) together show that $0 \leq a < 1$.

Answer $a \in [0, 1)$.

Remark Could a graphing calculator be helpful here? For this problem, it is difficult to find an exact answer on a calculator. However, graphing a function $Y_1 = \log(a\cos 2x - (1 + a^2 - \cos^2)\sin x + 4 - a)/\log 5 - 1$ we can watch for what values of parameter a it has a negative value (Figures 4.7, 4.8, 4.9, 4.10, 4.11, and 4.12).

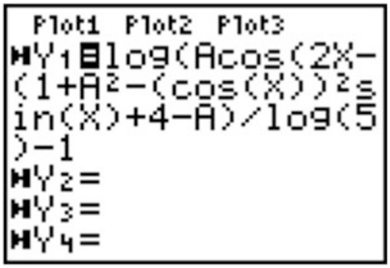


Figure 4.7 Problem 177

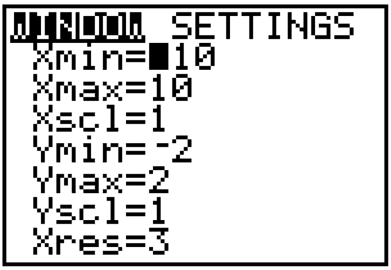


Figure 4.8 Setting for Problem 177

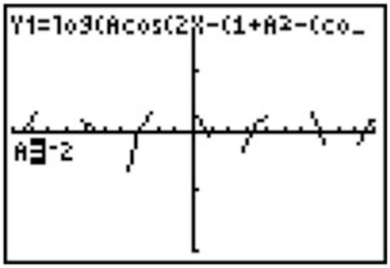


Figure 4.9 Graph at $A = -2$

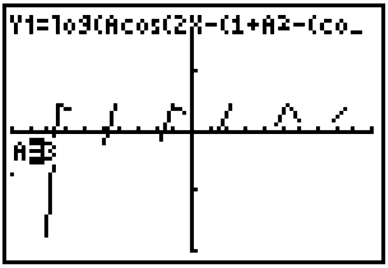


Figure 4.10 Graph at $A = 3$

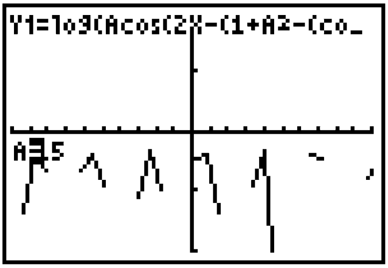


Figure 4.11 Graph at $A = 0.5$

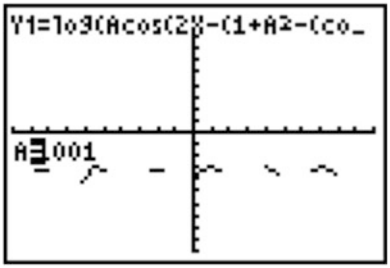


Figure 4.12 Graph at $A = 0.001$

We need to show that our entire function $Y1$ is below zero (the X-axis) only for a parameter a from the interval [0, 1). Graphing $Y1$ for $a = -2$ (Figure 4.9), $a = 3$ (Figure 4.10), $a = 0.5$ (Figure 4.11), and $a = 0.001$ (Figure 4.12) we can see that our analytical solution was correct.

Problem 178 For all real values of a solve the logarithmic equation $(\log_2 3)^{\sqrt{x+a+2}} = (\log_9 4)^{\sqrt{x^2+a^2-6a-5}}$.

Solution: A calculator will not help us here. So let us try an analytical approach. First, we rewrite $\log_9 4$ as a logarithm to the base 2:

$$\log_9 4 = \frac{\log_2 4}{\log_2 9} = \frac{2}{2\log_2 3} = \frac{1}{\log_2 3}$$

Replacing $\log_9 4$ by $(\log_2 3)^{-1}$ in the original equation, we obtain

$$(\log_2 3)^{\sqrt{x+a+2}} = (\log_2 3)^{-\sqrt{x^2+a^2-6a-5}} \tag{4.56}$$

Equation (4.56) can be considered as an exponential equation with base $\log_2 3$. From properties of exponents we know that if the bases are the same then the powers must be the same. (Remember how we would solve the equation $4^x = 4^3$? We would set $x = 3$.) For (4.56) we have

$$\sqrt{x+a+2} = -\sqrt{x^2+a^2-6a-5} \tag{4.57}$$

Does it look hard? May be at first glance. What is interesting about this equation?

We have a square root of something equals "minus" another square root of something. As you know $\sqrt{} \geq 0$ always. So the expression on the left is always greater or equal to zero, and the expression on the right (because of the "−" sign) is always less than or equal to zero:

$$\underset{\geq 0}{\sqrt{M}} = -\underset{\leq 0}{\sqrt{N}}$$

Such an equation, whatever the values of M and N, can have solutions if and only if

$$M = N = 0 \tag{4.58}$$

Using (4.58) we can rewrite (4.57) in an equivalent form as a system of two equations:

$$\begin{cases} x + a + 2 = 0 \\ x^2 + a^2 - 6a - 5 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -a - 2 \\ 2a^2 - 2a - 1 = 0 \end{cases}$$

Let us solve the second equation of the system separately:

$$2a^2 - 2a - 1 = 0$$
$$a_{1,2} = \frac{1 \pm \sqrt{3}}{2}$$

Answer We obtain two values of parameter a for which condition (4.58) can be satisfied.

If $a = \frac{1+\sqrt{3}}{2}$, then from the first equation of the system we have $x = -a - 2 = \frac{-1-\sqrt{3}}{2} - 2 = \frac{-5-\sqrt{3}}{2}$.

If $a = \frac{1-\sqrt{3}}{2}$, then $x = \frac{\sqrt{3}-3}{2}$

If $a \neq \frac{1\pm\sqrt{3}}{2}$, then the given equation has no solutions.

We obtain two possible solutions of the equation for two different values of parameter a. The solutions are not obvious; we cannot just guess and try. Using properties of functions and common sense we have solved this nonstandard equation.

Now let us try to solve the next problem using the same logic.

Problem 179 Solve the equation $\sqrt{2\cos(x+a) - 1} = \sin 6x - 1$ for every value of parameter a from $a \in \left(-\frac{\pi}{2}, 0\right)$.

Solution: Notice that $\sqrt{2\cos(x+a) - 1} \geq 0$ for all real x and a, and $\sin 6x - 1 \leq 0$ for all real x, then

$$\underbrace{\sqrt{2\cos(x+a) - 1}}_{\geq 0} = \underbrace{\sin 6x - 1}_{\leq 0}$$

The equation has solutions if and only if both of its sides equal 0. This statement can be written as the system:

$$\begin{cases} 2\cos(x+a) - 1 = 0 \\ \sin 6x - 1 = 0 \\ -\frac{\pi}{2} < a < 0 \end{cases} \tag{4.59}$$

First, let us solve the second equation:

$$\sin 6x = 1$$
$$6x = \frac{\pi}{2} + 2\pi n$$
$$x = \frac{\pi}{12} + \frac{\pi n}{3}, \quad n \in Z.$$

Substituting this x into the first equation of system (4.59) let us try to find a:

$$\cos\left(\frac{\pi}{12}+\frac{\pi}{3}\cdot n+a\right)=\frac{1}{2}.$$
$$\frac{\pi}{12}+\frac{\pi}{3}\cdot n+a=\frac{\pi}{3}+2\pi k, n,k=0,\ \pm 1,\ \pm 2,\ \ldots \tag{4.60}$$

or

$$\frac{\pi}{12}+\frac{\pi}{3}\cdot n+a=-\frac{\pi}{3}+2\pi m, \quad n,m=0,\ \pm 1,\ \pm 2,\ \ldots \tag{4.61}$$

(4.60) gives us

$$a=\frac{\pi}{4}+2\pi k-\frac{\pi}{3}\cdot n, \quad n,k=0,\ \pm 1,\ \pm 2,\ \ldots \tag{4.62}$$

Solving (4.61) we obtain that

$$a=-\frac{5\pi}{12}+2\pi m-\frac{\pi n}{3}, \quad n,m=0,\ \pm 1,\ \pm 2,\ \ldots \tag{4.63}$$

However, we have to select only those a, which satisfy the inequality of (4.59) or in other words, select a from $\left(-\frac{\pi}{2},0\right)$. Because $2\pi k$ for any $k\in Z$ puts the value of a outside the interval, we can omit $2\pi k, 2\pi m$ terms in (4.62) and (4.63).

There are two values of a that satisfy all restrictions simultaneously: $a=-\frac{\pi}{12}, \quad a=-\frac{5\pi}{12}$.

Answer $a=-\frac{\pi}{12}, \quad a=-\frac{5\pi}{12}$.

Problem 180 Find the values of parameter a over $[1,\infty)$ that maximize the largest of two roots of the quadratic equation $x^2-6x+2ax+a-13=0$.

Solution: Most students try to solve this problem directly. They express the largest root of the equation in terms of a as $x(a)=3-a+\sqrt{a^2-7a+22}$ and start to investigate the maximum of the function. Try it, please. If you do not intend to use a calculator then this way is useless. However, using the TI Interact program on a TI 83–84 Plus we can graph a function:

$$y=x^2-6x+2ax+a-13 \quad \text{and} \quad a\in[1,\infty).$$

Graphing this function for different values of parameter a from $[1,\infty]$ ($a=1$ (Figures 4.13 and 4.14), $a=2$ (Figures 4.15 and 4.16), and $a=6$ (Figures 4.17 and 4.18)) we can see that the largest of two roots is maximized ($x=6$) at $a=1$.

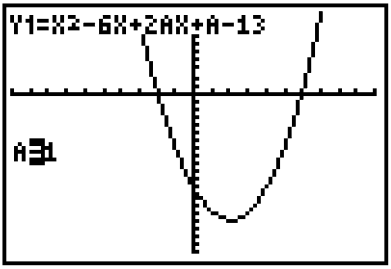


Figure 4.13 Problem 180 ($A = 1$)

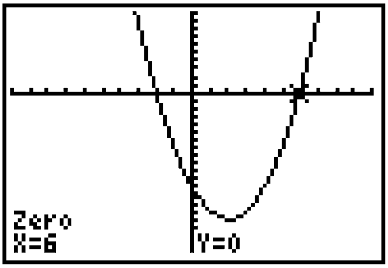


Figure 4.14 Positive zero at $A = 1$

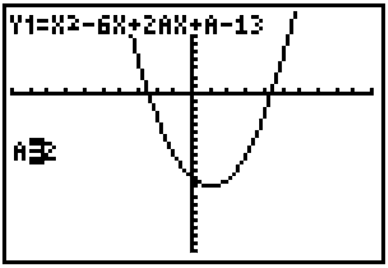


Figure 4.15 Problem 180 ($A = 2$)

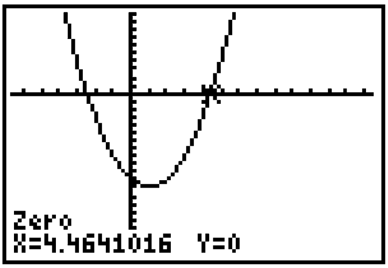


Figure 4.16 Positive zero at $A = 2$

This problem can be solved *algebraically* if we consider a function $a(x)$ that establishes the correspondence between each root x, not necessarily the largest, and that value of parameter a at which the given equation has this specific root x.

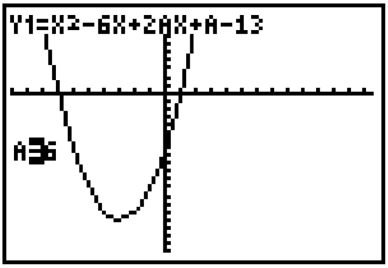


Figure 4.17 Problem 180 ($A = 6$)

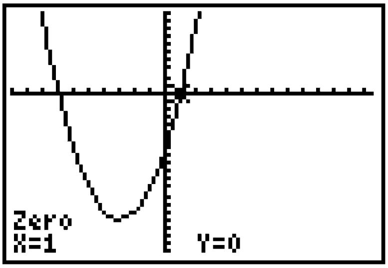


Figure 4.18 Positive zero at $A = 6$

Let us rewrite it as
$x^2 - 6x - 13 + a(2x + 1) = 0$ and solve it for a:

$$a(x) = \frac{-x^2 + 6x + 13}{2x + 1}$$

Note that $x = -1/2$ is not within the domain of $a(x)$, because if $x = -1/2$, then the original expression is not "0" for any a.

Check:

$$\left(-\frac{1}{2}\right)^2 - 6 \cdot \left(-\frac{1}{2}\right) + a \cdot \left(2 \cdot \left(-\frac{1}{2}\right) + 1\right) - 13 = \frac{1}{4} + 3 + a \cdot 0 - 13 = -9\frac{1}{4} \neq 0$$

What do we know about function $a(x)$?

We know that the range of $a(x)$ is $[1, \infty)$ or $a(x) \geq 1$.

But we are interested in its domain. Let us find the domain. We are going to find x such that the following inequality is true:

$$a(x) \geq 1 \quad \Leftrightarrow \quad \frac{-x^2 + 6x + 13}{2x + 1} \geq 1 \quad \Leftrightarrow$$
$$\frac{-x^2 + 4x + 12}{2x + 1} \geq 0$$

Multiplying both sides of the last inequality by (-1), and then factoring the numerator we obtain

$$\frac{x^2-4x-12}{2x+1} \le 0$$
$$\frac{(x-6)\,(x+2)}{2x+1} \le 0$$

Let us put all “critical” points on the number line in increasing order: -2, $-1/2$, and 6. We’ll put -2 and 6 as “closed” circles and $-1/2$ as “an open circle” (because $x=-1/2$ is excluded from the domain):

```
        −              +              −              +
------O------------O--------------O-----------
      −2          −1/2            6
```

We will check the sign of $f(x)=\dfrac{(x-6)(x+2)}{2x+1}$ on each interval and choose those intervals where $f(x)$ is negative. Thus, $a(x) \ge 1$ on $x \in (-\infty,\,-2] \cup \left[-\frac{1}{2}, 6\right]$.

We obtained ***the upper bound*** for all possible roots of the equation: $x=6$.

If $x=6$, then $a(6)=1$.

Now we have to investigate the last detail: whether $x=6$ is the largest of two possible roots for $a=1$. Replacing a by 1 into the given equation, we obtain

$$\begin{aligned}
&x^2-6x+2x+1-13=0\\
&x^2-4x-12=0\\
&(x-6)\,(x+2)=0\\
&x_1=6 \quad x_2=-2,\\
&-2<6 \quad \text{true}
\end{aligned}$$

Answer $a=1$.

You can argue that a calculator approach is faster, but solving a problem algebraically we move step by step, applying properties of functions, and we prove everything. When we solve a problem on a calculator sometimes we get a result, but cannot explain how we obtained it and why this or that answer appeared. If you are the kind of student who always wants to know why, then the algebraic method is for you. The criterion of competency in math is the ability to solve a problem analytically. Sometimes an analytical approach is shorter and more elegant. Since the purpose of this book is to give you such competency, I encourage you to do this.

Problem 181 Let $f(x)=\sqrt{x^2-4x+4}-3$ and $g(x)=\sqrt{x}-a$. Solve for x the inequality $f(g(x)) \le 0$.

Solution: Completing the square under the radical of the first function and using properties of an absolute value, we can rewrite $f(x)$ as

$$f(x) = \sqrt{(x-2)^2} - 3 = |x-2| - 3$$
$$g(x) = \sqrt{x} - a.$$

Next, we find the composition of $f(x)$ and $g(x)$ $((f \circ g)$.

By the condition of the problem this composition must be less than or equal to zero:

$$f(g(x)) \le 0 \Rightarrow |(\sqrt{x} - a) - 2| - 3 \le 0 \Rightarrow |\sqrt{x} - a - 2| \le 3$$
$$-3 \le \sqrt{x} - a - 2 \le 3$$
$$a - 1 \le \sqrt{x} \le a + 5$$

Let us consider three cases for this inequality:

If $a + 5 < 0\,(a < -5)$, then we have no solutions.

If $a - 1 < 0 \le a + 5$ $(-5 < a < 1)$, then $\sqrt{x} \le a + 5 \Leftrightarrow 0 \le x \le (a+5)^2$.

If $0 \le a - 1 (a \ge 1)$, then $a - 1 \le \sqrt{x} \le a + 5 \Leftrightarrow (a-1)^2 \le x \le (a+5)^2$.

Answer No solutions for $a < -5$; $x \in \left[0, (a+5)^2\right]$ if $-5 < a < 1$; $x \in \left[(a-1)^2, (a+5)^2\right]$ if $a \ge 1$.

Having solved this problem in general, for any value of a parameter a, you can always find some particular solution of the problem, and answer a question like the following:

If $a = 10$, what x will satisfy the given inequality $f(g(x)) \le 0$?

Analytically we obtained that if $a \ge 1$, then $x \in \left[(a-1)^2, (a+5)^2\right]$. If this is true, then this must be true for $a = 10$ as well, and the graph of function $y = f(g(x))$ must go below the X-axis only for $x \in \left[(10-1)^2, (10+5)^2\right]$ or $81 \le x \le 225$.

Let us check the answer on a TI 84 plus graphing calculator. First we make a composition of two functions on our graphing calculator (Figure 4.20). Here $Y_4 = g(x)$ and $Y_5 = f(g(x))$ (Figure 4.19).

We expect "negative" behavior between $x = 81$ and $x = 225$, then we need to set up an appropriate window for our calculator. Next we use **[2nd] [TRACE] (CALC)** buttons to find zeros of $f(g(x))$ (Figures 4.21, 4.22, 4.23, and 4.24).

As we expected, they are precisely 81 and 225 (Figures 4.25 and 4.26).

Problem 182 Find all values of a parameter a, such that the inequality $x^2 + 2|x - a| \ge a^2$ is true.

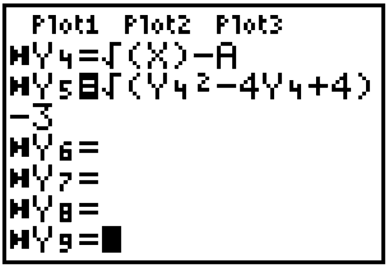


Figure 4.19 Graphing $f(g(x))$

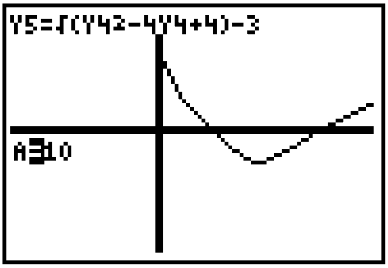


Figure 4.20 Problem 181 ($A = 10$)

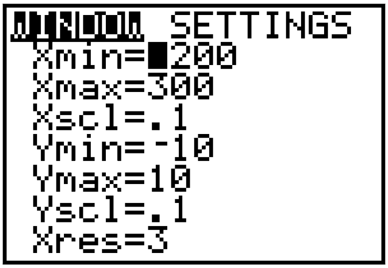


Figure 4.21 Setting for Problem 181

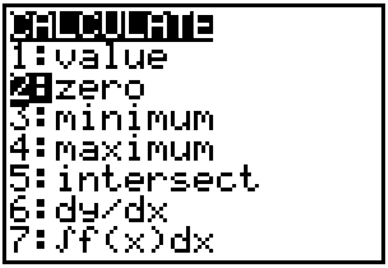


Figure 4.22 Finding X-intercept

Solution:

$$x^2 - a^2 + 2|x - a| \geq 0 \Leftrightarrow$$
$$(x - a)(x + a) + 2|x - a| \geq 0$$

Let us consider three different cases.

If $x = a$, then the obtained inequality is true because $0 \geq 0$

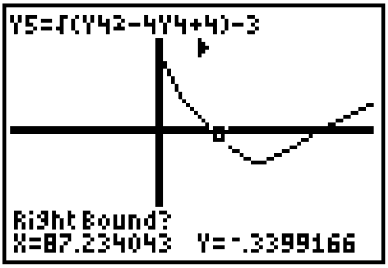


Figure 4.23 Right bound for a zero

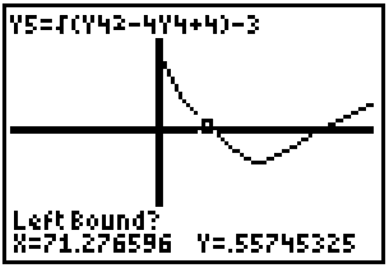


Figure 4.24 Left bound for a zero

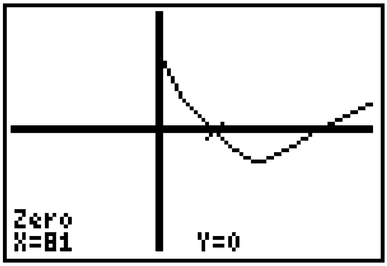


Figure 4.25 Solution $x = 81$

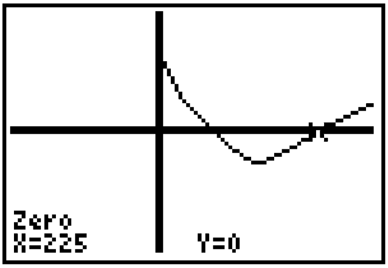


Figure 4.26 Solution $x = 225$

If $x \geq a$, then $(x-a)(x+a-2) \geq 0 \Rightarrow \begin{cases} x+a-2 \geq 0 \\ x \geq a \end{cases} \Rightarrow a \geq -1$

If $x < a$, then $(x-a)(x+a-2) \geq 0 \Rightarrow \begin{cases} x+a-2 \leq 0 \\ x < a \end{cases} \Rightarrow a \leq 1$

Answer $a \in [-1, 1]$.

Problem 183 Find all x for which $(4-2a)x^2+(13a-27)x+(33-13a) > 0$ for any value of parameter a from $1<a<3$.

Solution: Removing parentheses, we notice that the left side of the inequality is linear in a. This can be written as

$$\begin{aligned}&4x^2-2ax^2+13ax-27x+33-13a>0\\&a(-2x^2+13x-13)+(4x^2-27x+33)>0\end{aligned}$$

The last inequality can be written as

$$k(x)\cdot a+b(x)>0 \tag{4.64}$$

where $k(x)=-2x^2+13x-13$ and $b(x)=4x^2-27x+33$

If (4.64) is true, then the linear function $y=k(x)a+b(x)$ is above the X-axis on $a\in(1,3)$. Then at points $a=1$ and $a=3$ it must satisfy the following inequalities for some values of x:

$$\begin{aligned}&k(x)\cdot 1+b(x)\geq 0\\&k(x)\cdot 3+b(x)\geq 0\end{aligned} \tag{4.65}$$

Replacing $k(x)$ and $b(x)$ in (4.65) with their expressions from (4.64) we can find these x. Thus,

$$\begin{cases}(-2x^2+13x-13)+(4x^2-27x+33)\geq 0\\3\cdot(-2x^2+13x-13)+(4x^2-27x+33)\geq 0\end{cases}\Rightarrow$$
$$\begin{cases}(x-2)(x-5)\geq 0\\\left(x-\left(3-\sqrt{6}\right)\right)\left(x-\left(3+\sqrt{6}\right)\right)\leq 0\end{cases}\Rightarrow$$
$$\begin{cases}3-\sqrt{6}\leq x\leq 2\\5\leq x\leq 3+\sqrt{6}\end{cases}$$

Answer $x\in\left[3-\sqrt{6};2\right]\cup\left[5;3+\sqrt{6}\right]$.

Problem 184 Find all values of a, such that an inequality $x^2+4x+6a\cdot|x+2|+9a^2\leq 0$ has at most one solution.

Solution: Let us rewrite the inequality in a different form by completing a square on the left side:

$$|x+2| - 4 + 6a \cdot |x+2| + 9a^2 \leq 0$$
$$(|x+2| + 3a)^2 \leq 4$$

This inequality is equivalent to

$$-2 \leq |x+2| + 3a \leq 2$$

Subtracting $3a$ from both sides we obtain

$$-2 - 3a \leq |x+2| \leq 2 - 3a$$

This inequality will have at most one solution if and only if

$$2 - 3a \leq 0, \quad a \geq \frac{2}{3} \tag{4.66}$$

If (4.66) is true, then two cases are possible:

1. $|x+2| = 0$ and $x = -2$ (one solution). This happens if $a = 2/3$.
2. $|x+2| < 0$ and such an inequality has no solutions ($a > 2/3$).

Answer If $a \geq \frac{2}{3}$, then the inequality has at most one solution.

Problem 185 Find all values of a parameter a, for which the equation $a + \sqrt{6x - x^2 - 8} = 3 + \sqrt{1 + 2ax - a^2 - x^2}$ has precisely one solution.

Solution: Let us switch places of the square roots and then complete a square under each radical:

$$a - \sqrt{1 + 2ax - a^2 - x^2} = 3 - \sqrt{6x - x^2 - 8} \Rightarrow$$
$$a - \sqrt{1 - (x-a)^2} = 3 - \sqrt{1 - (x-3)^2}.$$

Next, we try to imagine graphs of the left and right sides of the above expression.

The graph of the right side is the lower half circle with center (3, 3) and radius 1.

The graph of the left side is the lower half of the circle with radius 1 as well but with the center (a, a).

If $a < 2$ or $a > 4$, these half circles do not have common points of intersection.

If $a = 3$, then the half circles coincide.

For all other values of a, the half circles have precisely one point of intersection (see Figure 4.27).

Answer $a \in [2, 3) \cup (3, 4]$.

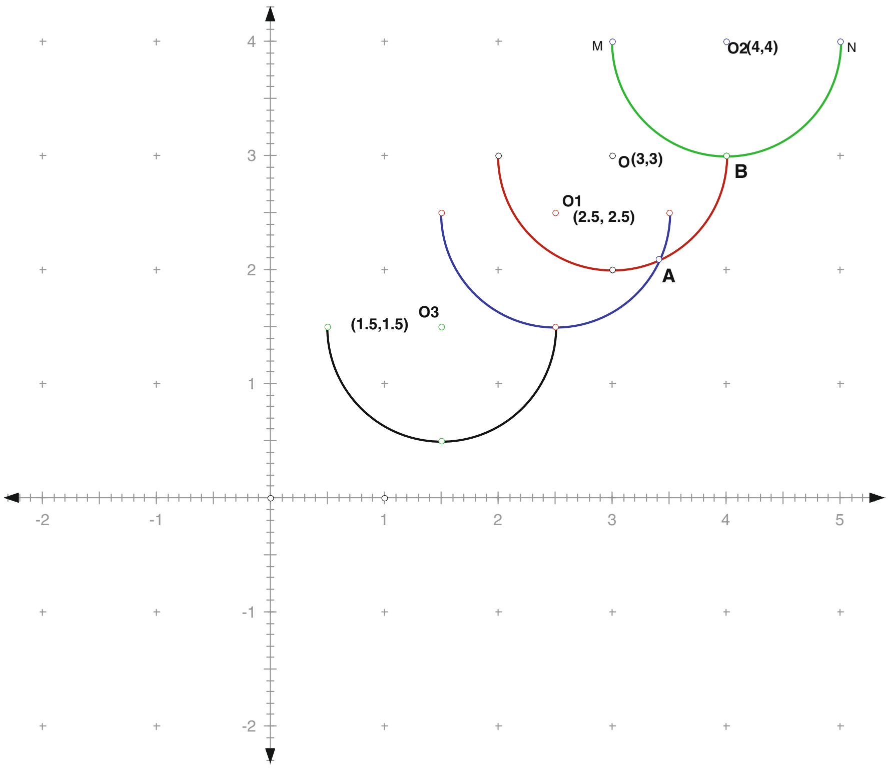


Figure 4.27 Sketch for Problem 185

Problem 186 Find all values of a parameter a for which the roots of the equation $\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=a$ exist and belong to the interval [2, 17].

Solution: Denoting a new variable, u, as $u=\sqrt{x-1}$, we can express x in terms of u: $x=u^2+1$ and rewrite the given equation as a new equation in u. Thus,

$$\sqrt{u^2-4u+4}+\sqrt{u^2-6u+9}=a$$
$$|u-2|+|u-3|=a \tag{4.67}$$

Next, if the roots of the given equation must be within $x\in[2,17]$, then the roots of (4.67) must be from $u\in[1,4]$.

Let us consider a function $y=y(u)$ in the restricted domain $1\le u\le 4$. Its range will give us the values of parameter a. Since $1\le y(u)\le 3$, $1\le a\le 3$.

Answer $1\le a\le 3$.

Problem 187 For what values of parameter a are all four roots of the equation $x^4 + (a-5)x^2 + (a+2)^2 = 0$ consecutive terms of an arithmetic progression?

Solution: Let us imagine a number line. Four numbers will be consecutive terms of some arithmetic progression only if they are $-3z$, $-z$, z, and $3z, z \neq 0, z > 0$. This means that quadratic equation

$$y^2 + (a-5)y + (a+2)^2 = 0$$

has the following roots:

$$y_1 = z^2 \quad \text{and} \quad y_2 = 9z^2 \tag{4.68}$$

Now by Vieta's Theorem we have

$$\begin{aligned} y_1 \cdot y_2 &= (a+2)^2 \\ y_1 + y_2 &= 5 - a \end{aligned} \tag{4.69}$$

Combining (4.68) and (4.69) we obtain

$$\begin{aligned} z^2 &= \frac{5-a}{10} \\ 9z^2 \cdot z^2 &= (a+2)^2 \end{aligned} \tag{4.70}$$

Eliminating z from (4.70) we obtain the following conditional equation for a:

$$\begin{aligned} a + 2 &= \frac{3(5-a)}{10}, \quad a \leq 5 \\ a &= -\frac{5}{13}. \end{aligned}$$

Answer $a = -5/13$.

4.6 Some Word Problems

4.6.1 Word Problems Involving Integers

It is not a secret that our students consider word problems as the most difficult. The reasons for this are different in each student's case. However, word problems usually are closely connected to real-life problems. Word problems in integers are also interesting because when solving them we need to use common sense in

order to select valid answers. For example, if one solved a problem and obtained the interval $3 < x < 7$ as a solution, where x is the number of apples in the basket, one must understand that the actual answer is 4, 5, and 6 only. Because only whole numbers of apples make sense. I want to start this section by giving you a problem that can be easily performed in any classroom and can serve as a "fun" introduction to more challenging problems on integers and divisibility.

Problem 188 A middle school teacher decided to introduce number theory to her 6th graders. Each student got a piece of paper and scissors. The teacher asked them to cut each paper into four pieces (not necessarily equal) and then take only one of the four pieces to cut into four more pieces. Following the second cutting, one of those four pieces is cut again into four pieces, and so on. After that, she wondered if her students could cut the original paper into 97 pieces. Some students continued to cut, but a few knew the answer right away. Do you know the answer too?

Solution: If one piece is broken into 4 pieces then the total number of pieces is increased by 3 pieces. At the beginning each student had one piece, and after the first cut each student had $1 + 3 = 4$ pieces. After the second cut, each student would have 7 pieces ($7 = 1 + 3 \cdot 2$). For example, after the third cut, a neat kid could have the cutting path represented by the red segments in Figure 4.28. Counting the number of pieces presented in this sketch, we obtain 10, which also can be written as $10 = 3 \cdot 3 + 1$.

If he/she continued, after the kth cut, they would have $(1 + 3k)$ pieces.

Because 97 divided by 3 gives a remainder of 1 ($97 = 1 + 3 \cdot 32$), it would be possible to get 97 pieces eventually.

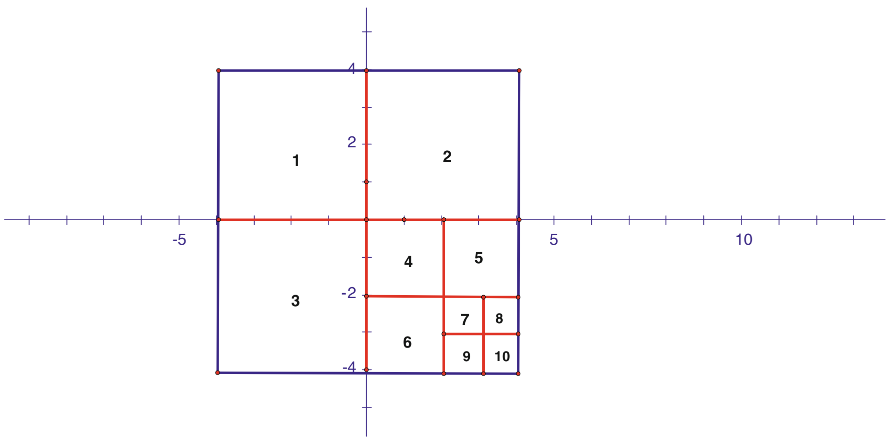


Figure 4.28 Sketch for Problem 188

Note It is interesting that the answer does not depend on how you cut a piece. It has to be four pieces, but there is no need for scissors; the four pieces can be unequal by size and even with fuzzy, not straight edges. (For example, one could rip the paper by hand.) It is clear that an ability to get n pieces depends on whether or not n divided by 3 gives a remainder of 1.

Problem 189 Three groups of fishermen caught 113 fish. The average number of fish per fisherman in the first group is 13, in the second 5, and in the third 4. How many fishermen were in each group if there were a total of 16 men?

Solution: We noticed that this problem is again a problem involving integers. We cannot have 5.5 fishermen or 7.3 fish. Let us introduce three variables corresponding to unknowns:

Let x be the number of fishermen in the first group.
y—the number of fishermen in the second group.
z—the number in the third group.
There were 16 men in total or

$$x + y + z = 16 \tag{4.71}$$

They caught 113 fish in total and using the conditions of the problem, we have the second equation:

$$13x + 5y + 4z = 113 \tag{4.72}$$

Usually two equations in three variables cannot have a unique solution. Using the fact that x, y, and z must be integers, we will find a solution. From (4.71) we can express z in terms of x and y and then find all integer solutions of (4.72):

$$z = 16 - y - x \tag{4.73}$$

then $13x + 5y + 4(16 - y - x) = 113$ or

$$y + 9x = 49 \tag{4.74}$$

Solving (4.74) for x and extracting an integer part of the quotient we obtain

$$x = \frac{49 - y}{9} = \frac{45 + 4 - y}{9} = 5 + \frac{4 - y}{9} \tag{4.75}$$

In order for x to be a natural number, $\frac{4-y}{9}$ must be an integer as well. Therefore $(4 - y)$ should be divisible by 9, so we can assume that

$$4 - y = 9n, \quad n \in Z \tag{4.76}$$

Substituting (4.76) into (4.73) and (4.75) gives us the following:

$$x = 5 + n \tag{4.77}$$
$$y = 4 - 9n \tag{4.78}$$
$$z = 7 + 8n \tag{4.79}$$

Using common sense we know that x, y, and z must be positive integers and can be written as a system of three inequalities:

$$\begin{cases} y = 4 - 9n > 0 \\ x = 5 + n > 0 \\ z = 7 + 8n > 0 \end{cases} \tag{4.80}$$

Solving each inequality of (4.80) for n we obtain

$$\begin{cases} n < \frac{4}{9} \\ n > -5 \\ n > -\frac{7}{8} \end{cases} \Leftrightarrow -\frac{7}{8} < n < \frac{4}{9} \tag{4.81}$$

There is only one integer n, $n = 0$, that satisfies inequality (4.81). Plugging $n = 0$ into (4.77), (4.78), and (4.79) we obtain

$$y = 4; x = 5; z = 7$$

Answer There were 5 fishermen in the first group, 4 in the second, and 7 in the third.

Problem 190 A fruit farmer wants to plant trees. He has fewer than 1000. If he plants them in rows, 37 trees per row, then there will be 8 trees remaining. If he plants 43 per row, there will be 11 remaining. How many trees does he have?

Solution: Let x be the number of trees. Using the conditions of the problem and assuming that the farmer has either n rows of 37 planted or m rows of 43 planted, we obtain the following:

$$x < 1000 \tag{4.82}$$

$$x = 37n + 8 \tag{4.83}$$
$$x = 43m + 11 \tag{4.84}$$

Combining (4.83) and (4.84) we have

$$37n + 8 = 43m + 11 \tag{4.85}$$

Presenting $43m$ as $(37m + 6m)$ we can factor the left and the right sides of (4.85):

$$\begin{aligned} 37n - 37m &= 6m + 3 \\ 37(n - m) &= 3(2m + 1) \end{aligned} \tag{4.86}$$

We want to find only integer solutions of (4.86) because we cannot plant half of a tree. Because 3 and 37 are primes, in order for (4.86) to have any integer solutions, $(n - m)$ must be divisible by 3 and $(2\,m + 1)$ by 37. But there are a lot of numbers that are multiples of 37:

37, 74,111, 148 ...

Let us use condition (4.82) that restricts the possible values of x. (The fruit farmer has less than 1000 trees.) Now combining (4.82) and (4.84) we obtain

$$x = 43m + 11 < 1000$$
$$43m < 989$$
$$m < 23$$

How can we use this inequality?

Recall that $(2\,m + 1)$ should be a multiple of 37?. But if $m < 23$, then

$$\begin{aligned} 2m + 1 &< 2 \cdot 23 + 1 = 47 \\ 2m + 1 &< 47 \end{aligned} \tag{4.87}$$

Looking at the last equation of (4.87) we can see that there is only one multiple of 37 that is less than 47: this is 37 itself. Therefore $2m + 1 = 37$, and $m = 18$. Replacing m = 18 into (4.84) we obtain

$$x = 43 \cdot 18 + 11 = 785$$

Answer The fruit farmer has 785 trees.

Problem 191 One box contains only red balls, and another only blue. The number of red balls is 15/19 of the number of blue balls. When 3/7 of the red balls and 2/5 of the blue are removed from the boxes, there are less than 1000 balls in the first box and greater than 1000 balls in the second. How many balls were originally in each box?

Solution: Let x be the original number of red balls in the first box. Let y be the original number of blue balls in the second box. Now we can write the following system of equations and inequalities:

$$\begin{cases} \frac{4}{7}x < 1000 \\ x = \frac{15}{19}y \\ \frac{3}{5}y > 1000 \end{cases} \tag{4.88}$$

Some of you could try to solve this system in a standard way and would obtain

$$\begin{cases} \frac{4}{7}\cdot\frac{15}{19}y < 1000 \\ x = \frac{15}{19}y \\ y > 1666\frac{2}{3} \end{cases} \Rightarrow \begin{cases} 1666\frac{2}{3} < y < 2216\frac{2}{3} \\ x = \frac{15}{19}y \end{cases} \tag{4.89}$$

Just a quick look at system (4.89) says that there are many numbers between 1666 and 2216, and also x and y must satisfy the second equation of the system. Let us try to find some nonstandard way of solving system (4.89). We will use properties of integers. If x and y exist and satisfy system (4.88), x must be divisible by 15 and y must be divisible by 19. In order that $\frac{4}{7}x$ and $\frac{3}{5}y$ be integers x must also be a multiple of 7 and y a multiple of 5. So we can represent

$$\begin{aligned} x &= 15 \times 7 \times x_1 \\ y &= 19 \times 5 \times y_1 \end{aligned} \tag{4.90}$$

where x_1, y_1 are some unknown integers.

Replacing these x and y into system (4.88) we obtain

$$\begin{cases} 7\cdot 15\cdot x_1 = \frac{15}{19}\cdot 5\cdot 19\cdot y_1 \\ \frac{4}{7}\cdot 7\cdot 15\cdot x_1 < 1000 \\ \frac{3}{5}\cdot 5\cdot 19\cdot y_1 > 1000 \\ x_1, y_1 \in N \end{cases} \Leftrightarrow \begin{cases} 7x_1 = 5y_1 \\ 3x_1 < 50 \\ 3\cdot 19\cdot y_1 > 1000 \\ x_1, y_1 \in N \end{cases} \tag{4.91}$$

In order for the first equation of (4.91) to have some integer solutions, x_1 must be a multiple of 5 and y_1 a multiple of 7. Let us introduce new variables (do a new substitution):

$$x_1 = 5 \cdot x_2$$
$$y_1 = 7 \cdot y_2$$

Now system (4.91) can be solved:

$$\begin{cases} 7 \cdot 5 \cdot x_2 = 5 \cdot 7 \cdot y_2 \\ 3 \cdot 5 \cdot x_2 < 50 \\ 3 \cdot 19 \cdot 7 \cdot y_2 > 1000 \\ x_2, y_2 \in N \end{cases} \Rightarrow \begin{cases} x_2 = y_2 \\ x_2 < \dfrac{10}{3} = 3\dfrac{1}{3} \\ y_2 > \dfrac{1000}{399} = 2\dfrac{202}{399} \\ x_2, y_2 \in N \end{cases} \tag{4.92}$$

From system (4.92) we notice that $2\,\dfrac{202}{399} < x_2 < 3\,\dfrac{1}{3}$ and $2\,\dfrac{202}{399} < y_2 < 3\,\dfrac{1}{3}$. Because there is only one integer $x_2 = y_2 = 3$, then

$$x = 7 \cdot 15 \cdot x_1 = 7 \cdot 15 \cdot 5 \cdot x_2 = 7 \cdot 15 \cdot 5 \cdot 3 = 1575$$
$$y = 5 \cdot 19 \cdot y_1 = 5 \cdot 19 \cdot 7 \cdot y_2 = 5 \cdot 19 \cdot 7 \cdot 3 = 1995$$

Answer There are 1575 red balls and 1995 blue balls.

Problem 192 There are 600 more applicants to the University from high school students than from people who worked full time. There are 5 times as many ladies among the high school students than ladies among the non-students. And there are n times as many men among the high school graduates than there are in the group of non-students, such that $6 \leq n \leq 12$ (where n is an integer). Find the total number of college applicants if there are 20 more men than women among the non-students.

Solution: In my elementary number theory class, my students always start by using the finite value of the parameter n, between 6 and 12, and then they try to decide which one fits the condition of the problem. However, this way is not as good as it seems to be. There are 7 integer values possible for n between 6 and 12, and you would solve seven similar problems and it could take a while . . ..

Let us find a general approach to this problem. Analysis of the conditions shows that in order to translate the problem into a mathematical language, it is sufficient, besides n, to introduce just one other variable, say x, as a number of some subset of the applicants. Which one? Recalling the previous problem maybe we will have to verify divisibility of the value of x by some other number, and then replace x by some "smaller" integer, we conclude that the better choice is to introduce x as the number of female non-students. Then the number of any other type of applicants is going to be a positive integer as well. Of course, we assume that x is a natural number. If x is the number of female non-student applicants, then $(x+20)$ is

the number of male non-student applicants, and the total of non-student applicants will be

$$(x + x + 20) \tag{4.93}$$

$(5x)$ will be the number of female high school students, and the total number of applicants from high school will be represented by

$$5x + n(x + 20) \tag{4.94}$$

By the condition of the problem expression (4.93) is 600 less than expression (4.94). Thus, we have the equation

$$2x + 20 + 600 = 5x + n(x + 20) \tag{4.95}$$

Solving (4.95) for x we obtain

$$x = \frac{620 - 20n}{n + 3} \tag{4.96}$$

Recalling that x must be a positive integer, we rewrite the right part of (4.96) as

$$x = \frac{620 - 20(n + 3) + 20 \cdot 3}{n + 3} = \frac{680}{n + 3} - 20.$$

In order that x be a natural number, the number 680 must be divisible by $(n + 3)$. A prime factorization of 680 gives $680 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 17$.

If $6 \le n \le 12 \Rightarrow 9 \le n + 3 \le 15$, then n can be only 9,10,11,12,13, 14, or 15. Notice that only $n + 3 = 10$ is a factor of 680. This means that $n = 7$. Then $x = \frac{680}{10} - 20 = 48$ and the total number of applicants is

$$2x + 20 + 5x + 7(x + 20) = 832.$$

Answer 832 applicants.

Problem 193 Three ranchers came to the Fort Worth Stock Show to sell their yearling heifers. The first rancher brought 10 heifers, the second 16, and third 26. On the first day every rancher sold some of his heifers. Moreover, all ranchers sold their heifers at the same price, one that had not changed during the entire first day. On the second day the price for heifers went down and all three ranchers in fear of further reductions in price sold all their remaining heifers at a reduced price per heifer. What was the price per heifer on the first day and on the second day if each rancher took home $3500?

Solution: Let us introduce three variables x, y, and z as a number of heifers sold by the first, second, and third ranchers, respectively, on the first day. Because the first rancher originally had 10 heifers, the second 16, and the third 26, on the second day they would sell $(10 - x)$, $(16 - y)$ and $(26 - z)$ heifers, respectively. Let us create a table.

Introducing two additional variables: t as the price for heifers on the first day of the Stock Show and p as the price for heifers on the second day, and using the conditions of the problem, we obtain the following system of three equations in five variables, x, y, z, t, and p:

$$\begin{cases} xt + (10 - x)p = 3500, & 1 \leq x \leq 9 \\ yt + (16 - y)p = 3500, & 1 \leq y \leq 15 \\ zt + (26 - z)p = 3500, & 1 \leq z \leq 25 \end{cases} \tag{4.97}$$

Combining like terms in each equation, we obtain

$$\begin{cases} x(t - p) + 10p = 3500 \\ y(t - p) + 16p = 3500 \\ z(t - p) + 26p = 3500 \end{cases} \tag{4.98}$$

Subtracting the last equation from the first and from the second we have

$$\begin{cases} (x - z)(t - p) = 16p \\ (y - z)(t - p) = 10p \end{cases} \tag{4.99}$$

Dividing these two equations gives us the equality:

$$\frac{x - z}{y - z} = \frac{16}{10}$$

or

$$(x - z) \cdot 5 = (y - z) \cdot 8 \tag{4.100}$$

Using our previous experience of solving equations like (4.100) for integers x, y, and z we conclude that $(x - z)$ should be divisible by 8 and $(y - z)$ by 5. These can be written as

Table 4.2 Problem 193

	Heifers brought	Heifers sold on the first day	Sold on the second day
Rancher 1	10	x	$10 - x$
Rancher 2	16	y	$16 - y$
Rancher 3	26	z	$26 - z$

$$\begin{cases} x - z = 8n, & n \in N, \quad 1 \leq x \leq 9 \\ y - z = 5m, & m \in N, \quad 1 \leq y \leq 15 \\ 1 \leq z \leq 25 \end{cases}$$

Because $x > z$, then $x - z = 8$ only. On the other hand, from (4.100) we can see that $y - z = 5$ only. If $x - z = 8$ and $y - z = 5$, then $x = 9$, $z = 1$, and $y = 6$. It means that on the first day the first rancher sold 9 heifers, the second rancher sold 6 heifers, and the third rancher sold only one heifer.

Knowing x, y, and z we can easily find t and p from equations of (4.99):

$$8(t - p) = 16p$$

$$5(t - p) = 10p, \text{ then } t - p = 2p \text{ or}$$

$$t = 3p \tag{4.101}$$

Using the first equation of system (4.98) and replacing t by $3p$ from (4.101) we obtain

$$\begin{aligned} &9 \cdot 2p + 10p = 3500 \\ &28p = 3500 \\ &p = 125 \\ &t = 3 \cdot 125 = 375 \end{aligned}$$

Answer On the first day the price per heifer was \$375 and on the second day \$125.

Problem 194 Today in one hospital the average age of doctors and patients together is 40 years. The average age of the doctors is 35 and the average age of the patients is 50. Are there more doctors or patients? How many times more?

Solution: Assume that d is the number of doctors in the hospital and p is the number of patients. Also assume that $p = k \cdot d$. Then the total age of the doctors is $35d$ and the patients is $50kd$. Then the quantity $35d + 50kd$ will represent the doctors and patients together and the following is true:

$$\begin{aligned} &35d + 50kd = 40(d + kd) \\ &5d(7 + 10k) = 5d(8 + 8k) \\ &2k = 1 \\ &k = \frac{1}{2} \end{aligned}$$

Answer There are twice as many doctors as patients.

Problem 195 An American tourist came to Moscow and was surprised at how difficult it was to pay for goods in Russian currency. Russians have kopeks and rubles. 100 kopeks is 1 ruble. It sounds similar to a dollar and cents but in Russia there are 1 kop., 2 kop., 3 kop., 5 kop., 15 kop., 20 kop., and 50 kop. coins (compare with US coins for 1, 5, 10, and 25 cents). The tourist put together in a plate some number (less than 15) of 3 kopeks and 5 kopeks coins and obtained 53 kopeks. He noticed that if in this set he replaces all 3 kopeks coins by 5 kopeks coins and all 5 kopeks coins by 3 kopeks coins, then the amount of money will decrease but not more than 1.5 times. How many 3 kopeks coins did our tourist originally have?

Solution: Let n be the initial number of 3 kopeks coins. Let m be the initial number of 5 kopeks coins. Then

$$3 \cdot n + 5 \cdot m = 53$$

Solving this equation for n we obtain

$$n = \frac{53 - 5m}{3} \tag{4.102}$$

In order for n to be a positive integer $(53 - 5\,m)$ must be a positive multiple of 3. There are four such opportunities for m: $m_1 = 1, m_2 = 4, m_3 = 7, m_4 = 10$. Solving (4.102) for each m we obtain four possible ordered pairs:

$$(m, n) : \{(1, 16), (4, 11), (7, 6) \text{ and } (10, 1)\} \tag{4.103}$$

Only one of (4.103) will satisfy the problem condition:

$$\begin{cases} n + m < 15 \\ 1.5(5n + 3m) \geq 53 \end{cases} \tag{4.104}$$

Checking all possible ordered pairs (4.103) we notice that only (7, 6) $m = 7$ and $n = 6$ satisfy (4.104).

Answer Our tourist had six 3 kopeks coins and seven 5 kopeks coins.

4.6.2 Other Word Problems

There are so many different types of word problems that we would need to write an entire book devoted only to them, and we would still not describe all possible problems. Earlier in this chapter you solved two interesting problems on a **mixture**

(Problems 150 and 152). In this section we will go over a few interesting problems and will give you some good ideas for your own search for a beautiful solution.

Some word problems are about **work**, for example, problems about two or more pipes filling a swimming pool or problems about two or more painters painting the walls in a house. Usually the entire job is considered to be 1. If this job can be done by n workers, and if the kth worker can have the entire job finished by himself or herself in t_k hours, then his or her production rate is $\frac{1}{t_k}$ work per hour. Then all n workers working together will finish the job in

$$t = \frac{1}{\frac{1}{t_1} + \frac{1}{t_2} + \ldots + \frac{1}{t_n}} \text{hours}$$

Let us solve the following problem.

Problem 196 Three painters working together for one hour painted 7/10 of the house. It is known that each painter alone would be able to paint the house in an integer number of hours. Each of the painters has a different rate of painting. In how many hours could each of the three painters working alone paint the entire house?

Solution: Let n, k, and m be the number of hours needed to paint the house for the first, second, and third painter, respectively. Assume that n, k, and m are distinct natural numbers satisfying the inequality:

$$1 < n < k < m \tag{4.105}$$

and then $\frac{1}{n}, \frac{1}{k}, \frac{1}{m}$ their production rates per hour will satisfy

$$\frac{1}{m} < \frac{1}{k} < \frac{1}{n} \tag{4.106}$$

From the condition of the problem we obtain that

$$\frac{1}{m} + \frac{1}{k} + \frac{1}{n} = \frac{7}{10} \tag{4.107}$$

Using inequality (4.106) we can state that

$$\frac{1}{m} + \frac{1}{k} + \frac{1}{n} = \frac{7}{10} < \frac{3}{n}$$

Or that

$$7n < 30 \Rightarrow n < \frac{30}{7}.$$

Because n is a natural number greater than 1, we obtain that $n = 2, 3, 4$:

$$2 \leq n \leq 4$$

Let us show that $n = 4$ is not a possible solution to this problem.

Substituting $n = 4$ into (4.107) we obtain the following true equality:

$$\frac{1}{m} + \frac{1}{k} + \frac{1}{4} = \frac{7}{10}$$
$$\frac{1}{m} + \frac{1}{k} = \frac{9}{20}$$

Because of (4.106) we can state that

$\frac{1}{m} + \frac{1}{k} < \frac{2}{k}$ and we will obtain the restriction on variable k:

$$\frac{9}{20} < \frac{2}{k}$$
$$9k < 40$$
$$k < \frac{40}{9}$$

From this we obtain possible values for k:

$$k = 2, 3, 4$$

That contradicts our assumption (4.105) and the fact that $k > n = 4$.

If $n = 3$, using a similar argument we will find restrictions on k:

$$\frac{1}{m} + \frac{1}{k} + \frac{1}{3} = \frac{7}{10}$$
$$\frac{1}{m} + \frac{1}{k} = \frac{11}{30}$$
$$\frac{11}{30} < \frac{2}{k}$$
$$11k < 60.$$
$$\begin{cases} k < \frac{60}{11} \\ k > n = 3 \end{cases}$$
$$k = \{4, 5\}$$

We obtain a solution $n = 3$, $k = 5$, and $m = 6$.

If $n = 2$ and using the same chain of arguments we obtain that

$$\frac{1}{m}+\frac{1}{k}+\frac{1}{2}=\frac{7}{10}$$
$$\frac{1}{m}+\frac{1}{k}=\frac{1}{5}$$
$$\frac{1}{5}<\frac{2}{k}$$
$$k<10.$$
$$\begin{cases} k<10 \\ k>n=2 \end{cases}$$
$$k=\{3,4,5,6,7,8,9\}$$

The second solution is $n = 2$, $k = 6$, and $m = 30$.

Answer $(n,k,m):\{(2,6,30),(3,5,6)\}$.

Additionally, there are many problems about **motion**. If a person, an animal, a car, a plane, etc. maintains the same speed, then the distance equals speed times time:

$$d=v\cdot t$$

If speed is not constant, and it is different on different time intervals, then the average speed equals the total distance divided by the total travel time. Please recall a speeding ticket example, discussed in Section 1.3 of Chapter 1.

Problem 197 Peter lives near a bus stop A. Bus stops A, B, C, and D are on the same street. Peter walks for exercise every weekend. He starts at A with a speed of 5 km per hour and goes to D. Reaching D he turns back and goes to B. Walking this route (A–D–B) requires 5 h. At B Peter takes a bus and goes home. It is known that he can cover the distance between A and C in 3 h. The distances between A and B, B and C, and C and D form a geometric sequence in the given order. Find the distance between B and C.

Solution: Usually it is a good idea to draw a picture of the problem. Bus stops A, B, C, and D are on the same street. It means that we can draw them as points on the same line; A and D will be the end points of the segment and B and C between them in the order A–B–C–D (Figure 4.29).

Because our unknown is the distance between B and C it seems obvious to introduce three variables x, y, and z as distances between A and B, B and C, and C and D, respectively. Using the condition of the problem and recalling that

Figure 4.29 Sketch for Problem 197

distance = *speed* × *time* we write two equations:

$$x + y + z + z + y = 5 \cdot 5 = 25 \quad \text{and} \quad x + y = 3 \cdot 5 = 15$$

Now we are going to write the last equation of the system. Because x, y, and z are consecutive terms of a geometric sequence, then $y^2 = xz$, and we can complete and solve the system as follows:

$$\begin{cases} x + 2y + 2z = 25 \\ x + y = 15 \\ y^2 = xz \end{cases}$$

$$\begin{cases} y + 2z = 10 \\ x = 15 - y \\ y^2 = xz \end{cases}$$

$$\begin{cases} z = \dfrac{10 - y}{2} \\ x = 15 - y \\ y^2 = \dfrac{(15 - y)(10 - y)}{2} \end{cases}$$

Subtracting the second equation from the first equation in the first system, we can eliminate variable x. (See the second system.) Then we express z and x in terms of y and put them into the third equation of the last system. Let us solve the last equation for y. Multiplying both sides by 2 we have

$$\begin{aligned} &2y^2 = 150 - 25y + y^2 \\ &y^2 + 25y - 150 = 0 \\ &y_1 = 5, \quad y_2 = -30 \end{aligned}$$

Because y is a distance, it has to be positive, so we choose $y = 5$.

Answer The distance between B and C is 5 km.

Finally, the last three word problems are not of any specific type. They involve a variety of different topics and ideas including knowledge of the properties of the natural numbers and number theory. Sometimes a problem looks like, for example, a "problem on work" but has nothing to do with the rate of production and is not such a problem at all. Only experience in solving many word problems of different types can make you an expert. I hope that your experience will be successful.

Problem 198 A boy was drinking a cup of tea with sugar. He put three spoons in one cup, dissolved it, drank 2/3 of the cup, then added one spoon of sugar, and filled the whole cup with the hot water. After dissolving the sugar in it and drinking 1/3 of the amount in the cup, the boy decided that the tea was not sweet enough. How much sugar should be added to the cup in order to make the tea as sweet as it was at the beginning?

Solution: In order to solve this problem we do not need any variables. First, 3 spoons of sugar per 1 cup is equivalent to 2 spoons of sugar per 2/3 of a cup or 1 spoon of sugar per 1/3 of the cup. Because the boy drank 2/3 of the sweet tea, there was exactly 1 spoon of sugar in 1/3 of the cup remaining. When he added 1 spoon of sugar to it, he had 2 spoons of sugar per 1/3 of cup. When he added hot water to it to fill the cup, then he had 2 spoons of sugar per one cup of hot water. The boy understood that the tea was not sweet enough when he drank 1/3 of the tea. At that moment he had $2(2/3) = 4/3$ spoons of sugar in 2/3 of the cup. Remember that he liked his original tea (2 spoons of sugar per 2/3 of the cup). Then the boy would have to add $\boldsymbol{x}$ spoons of sugar to it: $4/3 + x = 2$, $x = 2/3$ of a spoon of sugar.

Answer 2/3 of a spoon of sugar.

Problem 199 At night seven artists, in a certain order, painted a white wall, each with their own paint and color. Each artist painted $k\%$ of the wall without seeing what the previous artists painted. If any piece of the wall was painted by all 7 colors together, then it would again become white. For what integers k is there a warranty of the existence of at least one white piece on the wall?

Solution: Let 1 be the area of the wall. If each artist painted part of the wall with the area $s < 1/7$, then there would be an unpainted piece of the wall.

Since $14.2 < 100/7 < 14.3$ then the following integers 0, 1, 2, 3, …, 14 would work.

On the other hand, if $s > 6/7$, then each artist does not paint the area less than 1/7, then the union of all unpainted pieces by at least one of the artists has the area less than 1. Therefore, the remaining part of the positive area has all 7 colors. So the second part of the answer can be obtained as

$k > [6 \times 100/7] = 85$, so $k = 86, 87, 88,\ldots 99, 100$

Next we will consider the case $1/7 < s < 6/7$ and $k = 15, 16, 17, \ldots 84, 85$. Let us show that these values of k do not guarantee an existence of at least one white piece on the wall. We will demonstrate it for the following type of painting. This is such a painting of the wall that each point of the wall is painted but by less than 6 artists. Imagine the square with area 1 as a lateral surface of some cylinder. For any value of s from the considered interval the first artist would paint a part of the lateral surface with area s (shaded), then the next artist would paint a surface of area s that has common side with the previous one, etc. Finally, the entire surface would be painted but there will be no point at which all 7 colors meet.

Answer $k = \{1, 2,\ldots 14\}$ and $k = \{86, 87,\ldots 99, 100\}$.

Problem 200 Paul marked a rectangle with $m \times n$ cells on a piece of graph paper, such that m and n are relatively prime and $m < n$. The diagonal of this rectangle does not intersect precisely 116 "cells." Find all possible values of m and n.

Solution: Because m and n are relatively prime then the diagonal of the rectangle cannot go through the vertices of the interior cells. In fact, if it is not true then there will be a rectangle of a smaller size (made out of interior cells of the rectangle) such that its diagonal is part of the diagonal of the given rectangle. Let m_1 and n_1 be the width and length, respectively, of the biggest of such rectangles. Then obviously there exists such a number $k \in N, k > 1, m = km_1, n = k = kn_1$ that contradicts the condition. In the case when the diagonal of a rectangle does not contain vertices of the interior cells, then the number of the cells it intersects is one more than the sum of the vertical and horizontal lines going through the rectangle. Indeed, considering for example the diagonal going through the left lower vertex (angle) into the right upper vertex (angle) of the rectangle we can associate with each intersecting cell (except the upper right one) a unique point at which the diagonal intersects either a right or a left boundary of the cell. Note that in this manner each of the vertical and horizontal lines inside that rectangle will be intersected only once.

Therefore, the number of the intersecting cells is $(m-1)+(n-1)+1 = m+n-1$.

Then from the condition of the problem, we will obtain the equation that can be easily solved:

$$mn - 116 = m + n - 1$$
$$(m-1)(n-1) = 116$$

Because $m < n$, then the ordered pair $(m-1, n-1)$ can be selected out of the three possible answers: (1,116), (2,58), and (4,29).

However, the last one does not lead to a correct answer.

Answer (2;117) and (3;59).

4.7 Homework on Chapter 4

1. Find all values of a $(a < -4)$, for which the smallest root of the equation $x^2 + ax - 3x - 2a - 2 = 0$ has the minimal value.

   Solution: Let us solve the given equation for $a(x)$:

$$x^2 - 3x - 2 + a \cdot (x - 2) = 0$$
$$a(x) = \frac{-x^2 + 3x + 2}{x - 2}.$$

We can see that $x = 2$ is not in the domain of $a(x)$ because $2^2 - 3 \cdot 2 - 2 + a \cdot 0 \neq 0$.

The range of $a(x)$ is $(-\infty, -4]$.:

$$\frac{-x^2 + 3x + 2}{x - 2} \leq -4$$
$$\frac{x^2 - 7x + 6}{x - 2} \geq 0$$
$$\frac{(x - 1)(x - 6)}{x - 2} \geq 0$$

Using the intervals method, we can conclude the following:

$$a(x) \leq -4 \quad \text{if} \quad x \in [1, 2) \cup [6, \infty).$$

Moreover, $x = 1$ is the smallest value of $\boldsymbol{x}$ at which $a(x) \leq -4$. If $a = -4$

$$x^2 - 7x + 6 = 0$$
$$x_1 = 6, \quad x_2 = 1$$

Answer: $x = 1$ is the smallest of the two roots when $a = -4$.

2. There are three alloys. The first alloy contains 30 % of Ni and 70 % of Cu. The second contains 10 % of Cu and 90 % of Mn. The third—15 % of Ni, 25 % of Cu, and 60 % of Mn. We need to prepare a new alloy containing 40 % Mn. What is the smallest and the largest percentage of Cu (copper) that can be contained in the new alloy?
Answer: 40 % and 43 %.
3. There are two vessels containing a mixture of water and sand. In the first vessel there is 1000 kg of the mixture and in the second 1960 kg of the mixture. Water was added in both vessels. After that the % content of the sand in the first vessel was reduced k times, and in the 2nd l times. It is known that $kl = 9 - k$. Find the minimum amount of water that could be added to both vessels together.
Answer: 3480 kg.
4. Find the maximum of the function $f(x) = 5 \sin 2x + 7 \cos 2x$.
Answer: $\sqrt{74}$.
5. Find the maximum of $f(x) = 3 - 2 \sin^2 2x - 2 \cos 2x$.
Answer: $x = \pm\frac{\pi}{6} + \pi n, \quad n \in Z$.
6. Solve the inequality: $\sin\frac{x}{2} + \cos\frac{x}{2} \leq \frac{\sin x - 3}{\sqrt{2}}$
Answer: $x = \frac{\pi(8k - 3)}{2}, \quad k \in Z$.

7. Find all solutions of the equation $\frac{1}{\sqrt{2}}\sin^2\left(x+\frac{\pi}{2}\right) + \sin 3x = \cos 3x - \sqrt{2}$ on the interval $x \in [-2\pi, 2\pi]$.
Answer: $-\dfrac{\pi}{12}, \quad 2\pi, \quad -\dfrac{\pi}{12}$
8. Solve the inequality: $(x^2 - 4x + 3)\log_{\frac{1}{\sqrt{2}}}(\cos^2\pi x + \cos x + 2\sin x) \geq 2$
Answer: $x = 2$.
9. Solve the system:

$$\begin{cases} x^2 + 2x\sin y + 1 = 0 \\ 8|x|y(x^2 + y^2) + \pi^3 + 4\pi = 0 \end{cases}$$

Answer: $\left(1, -\frac{\pi}{2}\right)$.
10. Solve the equation $\sin^2 x + 3x^2 \cdot \cos x + 3x^2 = 0$.
Answer: $0; \pi + 2\pi n, \quad n \in Z$.
11. Solve the equation $2(1 + \sin^2(x-1)) = 2^{2x-x^2}$.
Hint: Use boundedness of the functions on the right and left sides.
Answer: $x = 1$.
12. Find all triples (x, y, z) satisfying the equation

$$x^2 + 1 - 2x\sin\pi y + \sqrt{yz - 2z^2 - 64} = (41 - yz)(\cos 2\pi y + \cos\pi z)^2$$

Answer: $\left(1, \dfrac{513}{2}, 128\right), \left(-1, -\dfrac{513}{3}, -128\right)$.
13. Find all ordered pairs (x, y) that satisfy the system:

$$\begin{cases} 2^{|x^2-2x-3|-\log_2 3} = 3^{-y-4} \\ 4|y| - |y-1| + (y+3)^2 \leq 8 \end{cases}$$

Answer: $(-1, -3)$ and $(3, -3)$.
14. Prove that if $xy + yz + zx = 1$, then $x + y + z \geq 1$.
Hint: Assume that $x+y+z < 1$ and prove by contradiction.
Proof: Assume that $x+y+z < 1$, then the following must be true:

$$\begin{aligned} &(x+y+z)^2 < 1 \\ &x^2 + y^2 + z^2 + 2xy + 2xz + 2yz < 1 \\ &x^2 + y^2 + z^2 + 2(xy + yz + xz) < 1 \\ &x^2 + y^2 + z^2 + 2 < 1 \\ &x^2 + y^2 + z^2 < -1. \end{aligned}$$

We can see that the last inequality is false. Therefore, our assumption was wrong and $x + y + z \geq 1$.

15. Prove that for any three positive real numbers a, b, and c, the inequality $(a+b+c)\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$ is true. For what values of three variables does the equality hold?

Hint: Multiply two quantities and apply the inequality between arithmetic and geometric means.
Proof: Using distributive law and after multiplication the left side of the inequality can be written as

$$1+\frac{a}{b}+\frac{a}{c}+\frac{b}{a}+1+\frac{b}{c}+\frac{c}{a}+\frac{c}{b}+1.$$

If we rearrange the terms and apply Cauchy's inequality to each pair of reciprocals, we will obtain

$$\begin{aligned} 3+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{c}{b}+\frac{b}{c}\right) &\geq 3+2\sqrt{\left(\frac{a}{b}\cdot\frac{b}{a}\right)}+2\sqrt{\left(\frac{a}{c}\cdot\frac{c}{a}\right)} \\ &\quad +2\sqrt{\left(\frac{c}{b}\cdot\frac{b}{c}\right)} \\ &= 9. \end{aligned}$$

The proof is completed. The equality holds if $a=b=c$.

16. Prove that for any natural number n, the inequality $n^n > 1\cdot 3\cdot 5\cdot 7\cdot\ldots\cdot(2n-1)$ is true.
17. Prove that $\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{100^2}<\frac{99}{100}$.

    Hint: Use the inequality $\frac{1}{k^2}<\frac{1}{(k-1)k}$.

    Proof:

$$\begin{aligned} \frac{1}{2^2} &< \frac{1}{1\cdot 2}=1-\frac{1}{2} \\ \frac{1}{3^2} &< \frac{1}{2\cdot 3}=\frac{1}{2}-\frac{1}{3} \\ &\ldots \\ \frac{1}{100^2} &< \frac{1}{99\cdot 100}=\frac{1}{99}-\frac{1}{100} \end{aligned}$$

Adding left and right sides of all inequalities we obtain

$$\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{100^2}<1-\frac{1}{100}=\frac{99}{100}.$$

The proof is completed.

18. Solve the equation $2\cos\frac{x}{16}=2^x+2^{-x}$.

    Answer: $x=0$.
19. Solve the equation $\sqrt{1+\sin x}+\sqrt{1+\cos x}=\sqrt{4+2\sqrt{2}}$.

    Answer: $x=\frac{\pi}{4}+2\pi n, n\in Z$.

20. Find all real solutions of the inequality $\left|\tan^3 x\right| + \left|\cot^3 x\right| \le 2 - (x - \pi/4)^2$
Answer: $x = \frac{\pi}{4}$.

Find all values of parameter b for which the system $\begin{cases} 4y = 4b + 3 - x^3 + 2x \\ x^2 + y^2 = 2x \end{cases}$ has precisely two solutions.
Solution: After some manipulations, the given system can be written as

$$\begin{cases} y^2 - 4y + 3 = -4b \\ (x-1)^2 + y^2 = 1 \end{cases} \Leftrightarrow \begin{cases} (y-2)^2 = -4b + 1 & (A) \\ (x-1)^2 + y^2 = 1 & (B) \end{cases}$$

where curve (A) is a parabola, $f(y) = (y-2)^2 + 2b - 1$ with the vertex $(2, -4b+1)$, and curve (B) is a circle with center (1,0) and radius 1. We can see that the two curves will have two points of intersection if

$$\begin{aligned} &-1 < y < 1 \\ &f(-1) > -4b > f(1) \\ &0 < -4b < 8 \end{aligned}$$

This gives the final answer: $-2 < b < 0$.
Answer: $-2 < b < 0$.

21. What is greater 10! or 7^{10}? Do not use a calculator.
Hint: See Problem 156.
Answer: $7^{10} > 10!$

22. Find all functions $f(x)$, such that for any real x, the following relationship holds:

$$f(x) + x \cdot f(1-x) = 3x. \tag{4.108}$$

Answer: $f(x) = \dfrac{3x^2}{x^2 - x + 1}$.

Solution: If (4.108) is true for any x, then it must be true for $(1-x)$. Substituting $(1-x)$ for x in (4.108), we obtain

$$f(1-x) + (1-x) \cdot f(x) = 3(1-x) \tag{4.109}$$

Solving (4.108) and (4.109) together and by eliminating $f(1-x)$, we obtain

$$x(1-x) \cdot f(x) - f(x) = 3x \cdot (1-x) - 3x$$

or

$$f(x) = \frac{3x^2}{x^2 - x + 1}.$$

23. Solve the system $\begin{cases} x^2y^2 - 2x + y^2 = 0 \\ 2x^2 - 4x + 3 + y^2 = 0 \end{cases}$

Hint: Consider both equations as quadratic in variable x. Remember that a quadratic equation has a solution in real numbers if its discriminant is nonnegative.
Solution: Let us find the discriminant of the first equation:

$$y^2x^2 - 2x + y^2 = 0, \quad a = y^2, \quad b = -2, \quad x = y^2.$$

Because the coefficient of the linear term is even, then we will use D/4 formula:

$$\frac{D}{4} = 1^2 - y^4 = 1 - y^4 \geq 0.$$

On the other hand, for the second equation of the system we have

$$2x^2 - 4x + (3 + y^3) = 0, a = 2, b = -4, c = 1 + y^3$$
$$\frac{D}{4} = 2^2 - 2 \cdot (3 + y^3) = 2(-1 - y^3) \geq 0.$$

The condition that both discriminants be positive is equivalent to the system:
Then the corresponding $x = 1$. The answer is unique, it is the ordered pair $(x, y) = (1, -1)$.
Answer: $(1, -1)$.

24. For what values of a does the system $\begin{cases} 2x + y = a - 1 \\ 2xy = a^2 - 3a + 1 \\ 4x^2 + y^2 \leq -a^2 + 5a - 4 \end{cases}$
have a solution?
Answer: $a = 3$.
25. Given $a > 0, b > 0, c > 0$ and $a^2 + b^2 + c^2 = 1$. Prove that $a + b + c \leq \sqrt{3}$.
Hint: Use Cauchy-Bunyakovsky (CB) inequality.
Proof: The left side of the inequality to be proved can be rewritten as $a \cdot 1 + b \cdot 1 + c \cdot 1$. Applying the CB inequality to it and substituting the condition constrains $a^2 + b^2 + c^2 = 1$, we have

$$(a \cdot 1 + b \cdot 1 + c \cdot 1)^2 \leq \left(a^2 + b^2 + c^2\right)\left(1^2 + 1^2 + 1^2\right) = 3\left(a^2 + b^2 + c^2\right) = 3,$$

which leads us to the proof of the statement

$$a + b + c \leq \sqrt{3}.$$

26. Prove that for any real x, y, and z, the inequality is true: $x^2 + 2xy + 3y^2 + 2x + 6y \geq -3$.
Hint: Try to complete the squares on the left side.

Proof: The left-hand side can be rewritten as

$$\begin{aligned}
&\left\{x^2+2x(y+1)+(y+1)^2\right\}-(y+1)^2+3y^2+6y=\\
&(x+y+1)^2-y^2-2y-1+3y^2+6y+3-3=\\
&\underbrace{(x+y+1)^2}_{\geq 0}+\underbrace{2(y+1)^2}_{\geq 0}-3\geq -3.
\end{aligned}$$

27. Given $a\geq 0, b\geq 0, c\geq 0$. Prove that $\dfrac{2a}{b+c}+\dfrac{2b}{a+c}+\dfrac{2c}{a+b}\geq 3$.

Hint: Add 2 to each fraction on the left and put 2 and the fraction over the common denominator.
Proof: Adding 2 to each fraction we obtain new inequality to prove:

$$\begin{aligned}
&\frac{2a}{b+c}+2+\frac{2b}{a+c}+2+\frac{2c}{a+b}+2\geq 9\\
&\frac{2a+2b+2c}{b+c}+\frac{2b+2c+2a}{c+a}+\frac{2c+2a+2b}{a+b}\geq 9\\
&2(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\geq 9
\end{aligned}$$

It is convenient to denote $x=b+c,\quad y=c+a,\quad z=a+b\ \Rightarrow x+y+z=2(a+b+c)$.

And the left side of the last inequality will be rewritten as follows:

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Using distributive law, we will multiply two factors and rewrite the expression above as

$$1+\frac{x}{y}+\frac{x}{z}+\frac{y}{x}+1+\frac{y}{z}+\frac{z}{x}+\frac{z}{y}+1$$

Combining reciprocals together and after simplification, this expression becomes greater than or equal to 9:

$$3+\underbrace{\left(\frac{x}{y}+\frac{y}{x}\right)}_{\geq 2}+\underbrace{\left(\frac{x}{z}+\frac{z}{x}\right)}_{\geq 2}+\underbrace{\left(\frac{y}{z}+\frac{z}{y}\right)}_{\geq 2}\geq 3+6=9.$$

Therefore, the proof is completed.
Note: Above, for each parenthesis we used the inequality between the arithmetic and geometric means.

For example, $\frac{x}{y}+\frac{y}{x}\geq 2\sqrt{\frac{x}{y}\cdot\frac{y}{x}}=2$.

28. Solve the equation $\sqrt{1-x}+\sqrt{1+x}+\sqrt[4]{1-x^2}\ +\ \sqrt[4]{1+x^2}=4.$
Answer: $x=0$.
Hint: Apply Bernoulli's inequality to each term on the left-hand side.
Solution: Using Beroulli's inequality we have the following:

$$(1-x)^{\frac{1}{2}}\leq 1-\frac{1}{2}x$$
$$(1+x)^{\frac{1}{2}}\leq 1+\frac{1}{2}\,x$$
$$(1-x^2)^{\frac{1}{4}}\leq 1-\frac{1}{4}x^2$$
$$(1+x^2)^{\frac{1}{4}}\leq 1+\frac{1}{4}x^2$$

If we add the left and the right sides of these inequalities, we will obtain that the left side is less than or equal to 4. Therefore the function on the left of the given equation is bounded from the top by 4. In order to have a solution, the left side must take the value of 4. It occurs at $x=0$.

29. Find the minimal value of xy, where x and y satisfy the following system: $\begin{cases} x+y=3a-1 \\ x^2+y^2=4a^2-2a+2 \end{cases}$

Hint: Complete the square in the second equation. See Problem 177 of the book.
Answer: $-9/10$.
Solution: Completing the square in the second equation of the system and substituting $x+y=3a-1$ there we obtain

$$\begin{aligned}
&(3a-1)^2-4a^2+2a-2=2xy\\
&2xy=5a^2-4a-1\\
&2xy=5\left(a-\frac{2}{5}\right)^2-\frac{9}{5}\\
&xy=\frac{5}{2}\left(a-\frac{2}{5}\right)^2-\frac{9}{10}
\end{aligned}$$

It is easy to see that the minimum of xy is $-9/10$ and it occurs at $a=2/5$.

30. Solve the system $\begin{cases} x^2+y^2=0.5xyz \\ y^2+z^2=0.5xyz \\ z^2+x^2=0.5xyz \end{cases}.$
If we subtract the second and first, second and third, and third and the first equation we will obtain

$$\begin{cases} (x-z)(x+z)=0 \\ (y-z)(y+z)=0 \\ (y-x)(y+x)=0 \end{cases}$$

Because of the symmetry of this system, we notice that if (a,a,a) is a solution, then $(-a,a,-a),(-a,-a,a),(a,-a,-a)$ are also solutions. We get that

$$2a^2 = 0.5a^3$$
$$a = 0, a = 4$$

Answer: $(0,0,0),(4,4,4),(-4,-4,4),(4,-4,-4),(4,-4,-4)$.

31. Find the maximum and minimum of $\frac{y^2}{25}+\frac{w^2}{144}$ subject to system

$$\begin{cases} x^2+y^2+2x+4y-20=0 \\ z^2+w^2-2w-143=0 \\ xw+yz-x+w+2z-61 \geq 0 \end{cases}$$

Hint: See Problem 174.
Solution: The system will be rewritten as

$$\begin{cases} (x+1)^2+(y+2)^2=5^2 \\ (z)^2+(w-1)^2=12^2 \\ (x+1)(w-1)+z(y+2) \geq 60 \end{cases}$$

After substitution

$$\begin{aligned} &x+1 = 5\cos\varphi \\ &y+2 = 5\sin\varphi \\ &z = 12\cos\psi \\ &w-1 = 12\sin\psi, \quad \varphi,\psi \in [0,2\pi) \end{aligned}$$

The inequality will be written as follows: $60\sin(\varphi+\psi) \geq 60$.

That has solutions if and only if

$\psi = \frac{\pi}{2}-\varphi+2\pi n$, and then $\cos\psi = \sin\varphi, \quad \sin\psi = \cos\varphi$.

Now we see that $y = 5\sin\varphi - 2, \quad w = 12\cos\varphi + 1$. After substitution we obtain

$$\begin{aligned} \frac{y^2}{25}+\frac{w^2}{144} &= \frac{(5\sin\varphi-2)^2}{25}+\frac{(12\cos\varphi+1)^2}{144} \\ &= 1+\frac{4}{25}+\frac{1}{144}+\left(-\frac{4}{5}\sin\varphi+\frac{1}{6}\cos\varphi\right) \\ \mathrm{A}_{\max/\min} &= \frac{4201}{3600} \pm \frac{\sqrt{601}}{30} \end{aligned}$$

Answer: $\frac{4201}{3600} \pm \frac{\sqrt{601}}{30}$.

32. Solve the system of equations:

$$\begin{cases} x^2+xy+y^2=7 \\ x^2+xz+z^2=21 \\ y^2+yz+z^2=28 \end{cases}$$

Hint: Subtract the second and first, the third and second equations, and the third and first equations. Simplify, and apply difference of squares formula and factor out common factors:

$$\begin{cases} (z-y)(z+y+x)=14 \\ (y-x)(y+x+z)=7 \\ (z-x)(y+z+x)=14 \end{cases} \Rightarrow z=3y-2x \quad \text{etc.}$$

Answer:
$(x,y,z):\left\{(1,2,4),(-1,-2,-4),\left(-\sqrt{7},0,2\sqrt{7}\right),\left(\sqrt{7},0,-2\sqrt{7}\right)\right\}$.

33. Solve the system: $\begin{cases} y=2x^2-1 \\ z=2y^2-1 \\ x=2z^2-1 \end{cases}$.

Hint: Use trigonometric substitution, $x=\cos t,\quad 0\le t\le\pi$.
Answer:

$$\left[\begin{array}{l} (x,y,z)=\left(\cos\dfrac{2\pi k}{9},\ \cos\dfrac{4\pi k}{9},\ \cos\dfrac{8\pi k}{9}\right), k=0,1,2,3,4 \\ (x,y,z)=\left(\cos\dfrac{2\pi k}{7},\ \cos\dfrac{4\pi k}{7},\ \cos\dfrac{8\pi k}{7}\right), k=1,2,3. \end{array}\right.$$

34. Let a, b, and c be the sides of the triangle and h_a, h_b, h_c are the heights dropped to the sides, respectively. S is the area of triangle ABC. Prove that $\left(a^2+b^2+c^2\right)\cdot\left(h_a^2+h_b^2+h_c^2\right)\ge 36S^2$.
Proof: Because the area of the same triangle can be found in three different ways, we can start from the obvious equality
$\frac{ah_a}{2}+\frac{bh_b}{2}+\frac{ch_c}{2}=3S$, which can also be written as

$$(ah_a+bh_b+ch_c)^2=36S^2$$

Finally, we will apply the CB inequality to the left-hand side of the equality above and obtain
$(ah_a+bh_b+ch_c)^2=36S^2\le\left(a^2+b^2+c^2\right)\left(h_a^2+h_b^2+h_c^2\right)$, which proves the requested inequality.

35. At a classroom costume party, the average age of the b boys is g, and the average age of the g girls is b. If the average age of everyone at the party (all these boys and girls, plus their 42-year-old teacher) is $b+g$, what is the value of $b+g$?

Solution: Let us consider the average age in the group of g girls. If we add up the ages of g girls (let us say A) and divide it by the number of girls (g) we obtain b. On the other hand, if we add up the ages of all b boys (B) and divide it by the number of boys (b) we obtain g. This can be written as

$$\frac{A}{g} = b, \quad \frac{B}{b} = g \tag{4.110}$$

Then the average age of everyone at the party including the teacher will be

$$\frac{A + B + 42}{b + g + 1} = b + g \tag{4.111}$$

Replacing A and B from (4.110), we can rewrite (4.111) as

$$\frac{2gb + 42}{b + g + 1} = b + g$$

or multiplying both sides by $(b + g + 1)$ we obtain

$$2gb + 42 = (b + g)(b + g + 1) \tag{4.112}$$

We have from (4.112) that $42 = b^2 + g^2 + b + g$.

Multiplying both sides by 4 we have

$$\begin{aligned} 168 &= 4b^2 + 4g^2 + 4b + 4g \\ 168 + 2 &= \left(4b^2 + 4b + 1\right) + \left(4g^2 + 4g + 1\right) \end{aligned}$$

Completing the square, we have

$$\begin{aligned} &170 = (2b + 1)^2 + (2g + 1)^2 \\ &2b + 1 = 11, \quad 2g + 1 = 7 \Rightarrow b = 5, g = 3 \end{aligned}$$

or

$$2b + 1 = 7, \; 2g + 1 = 11 \Rightarrow b = 3, g = 5$$

Answer: $b + g = 8$.

36. A box has red, white, and blue balls. The number of blue balls is at least the number of white balls and at most 1/3 of the number of the red balls. The total number of white and blue balls is at least 55. What is the minimal possible number of red balls?

Solution: Let r, w, and b be the number of red, white, and blue balls, respectively. By the condition of the problem the following is valid:

$$\begin{cases} b \geq w \\ b \leq \dfrac{r}{3} \end{cases}$$

This can also be written as a double inequality:

$$w \leq b \leq \frac{r}{3}.$$

Adding b to all sides of the inequality, we obtain

$$w + b \leq 2b \leq \frac{r}{3} + b.$$

Using the information about the total number of blue and white balls and the second inequality of the system, we have the following:

$$55 \leq b + w \leq \frac{r}{3} + w \leq \frac{r}{3} + \frac{r}{3} = \frac{2r}{3}$$

This can be written as

$$55 \leq b + w \leq \frac{2r}{3}$$
$$2r \geq 165$$
$$r \geq 82.5$$

Because the number of the balls can be only a natural number, then $r = 83$ will be the first possible choice to check. If we substitute it into the first double inequality, we will obtain $w \leq b \leq 27$ but this contradicts the condition $b + w \geq 55$. Let us check the next natural number for r, $r = 84$; we obtain that

$$w \leq b \leq \frac{84}{3} = 28$$

And that $w = b = 28$ would give us the minimal number of balls in the box.
Answer: The box has 28 white, 28 blue, and 84 red balls.

37. A frustrated bond investor tears a bond into eight pieces. Then she continues and cuts one of the pieces again into eight pieces. If she continues, can she get 2016 pieces?
Hint: See Problem 188.
Answer: No, it is not possible.
Solution: If one piece is broken into 8 pieces then the number of pieces is increased by 7 pieces. At the beginning she had one piece, one bond, and after the first cut she had $1 + 7 = 8$ pieces. If she continued, she would get $(1 + 7\,k)$ pieces. Unfortunately, 2016 divided by 7 is divisible by 7 and cannot give us a remainder of 1. Therefore, the bond investor cannot tear the bond into 2016 pieces, but could cut it into 2017 pieces: $(2017 = 7 \cdot 288 + 1)$.

38. Using a red pipe the tank is filled in 3 h and using a blue pipe in 9 h. How soon will the tank be filled if both pipes are open?
Solution: The rate of filling the tank by the red pipe is 1/3 of the tank per hour and by the blue pipe is 1/9 of the tank per hour. Working together the two pipes have the following rate $\frac{1}{3}+\frac{1}{9}=\frac{4}{9}$ of the tank per hour. Then the entire tank will be filled in 9/4 h.
Answer: The tank will be filled in 2 h and 15 min.
39. Maria has 1 kg of 3.5 % salt solution and 1 kg of 0.5 % solution. Can she get 1.5 kg of 1.5 % by mixing two solutions? How can she do it?
Answer: Yes she can if she takes 0.5 kg of the first solution and 1 kg of the second solution.
Solution: If we mix x_1 kg of a solution (alloy, etc.) containing p_1 % of a substance A and x_2 kg containing p_2 % of A, then we will obtain (x_1+x_2) kg of new solution that will contain $\frac{p_1x_1+p_2x_2}{x_1+x_2}$ % of substance A.
Assume that she takes x kg of the first solution y kg of the second, then the following is true:

$$\begin{cases} \dfrac{3.5x+0.5y}{x+y}=1.5 \\ x+y=1.5 \end{cases} \Rightarrow x=0.5, y=1.5$$

40. A boat is going 10 miles downstream in a river and then against the current for a distance of 6 miles. The rate of current is 1 mile per hour. What is the speed of the boat itself (the speed it would have in still water) if the total trip time is between 3 and 4 h?
Solution: Let v be the boat speed then the total time of the trip can be expressed as

$$t=\frac{10}{v+1}+\frac{6}{v-1}.$$

It is clear that $v>1$; otherwise the boat would not be able to go against the stream.
Using this and the condition of the problem we obtain

$$3\le\frac{10}{v+1}+\frac{6}{v-1}\le 4$$

$$\begin{cases} 3(v^2-1)\le 16v-4\le 4(v^2-1) \\ 4v^2-16v\ge 0 \end{cases}$$

$$\frac{8-\sqrt{61}}{3}\le v\le\frac{8+\sqrt{61}}{3}$$

$$4\le v\le\frac{8+\sqrt{61}}{3}.$$

Answer: $4 \le v \le \dfrac{8+\sqrt{61}}{3}$ miles per hour.

42. An object is moving from point A to point B with a speed of 10 m/s for the first half of the distance and a speed of 15 m/s for the second half of the distance. What is the average speed on AB?

Solution: Let x be the half of the distance, so the total distance is $2x$. Then the object travelled the first half of the distance in $\dfrac{x}{10}$ seconds and the second half in $\dfrac{x}{15}$ seconds. The total travel time is $\dfrac{x}{10}+\dfrac{x}{15}=\dfrac{x}{6}$ seconds.

The average speed is $2x \div \dfrac{x}{6} = 12$ m/s.

Answer: 12 m/s.

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E. Grigorieva, *Methods of Solving Nonstandard Problems*,
DOI 10.1007/978-3-319-19887-3

Contest Problems for Further Reading

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DOI 10.1007/978-3-319-19887-3